Geometry of Matroids

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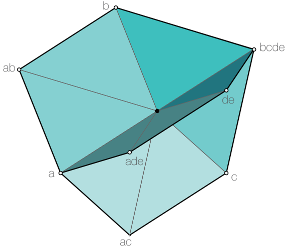
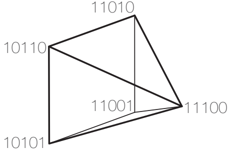
University of Waterloo, August 3, 2018





Summary.

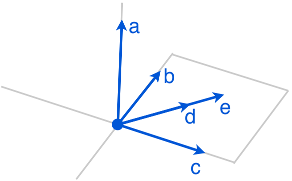
- Matroids are everywhere.
- There are many ways of thinnking about matroids.
- Geometry and matroid theory help each other.

Most of the work of mine that I will talk about is joint with Carly Klivans (06), Graham Denham + June Huh (17-18).

Matroids

Goal: Capture the combinatorial essence of independence.

E= set of vectors spanning $\mathbb{R}^d$.
$\mathcal{B}$ = collection of subsets of E which are bases of $\mathbb{R}^d$.



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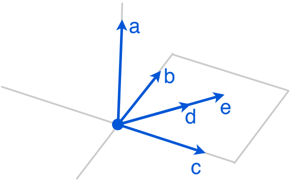
Properties:
(B1) $\mathcal{B} \neq \emptyset$
(B2) If $A, B \in \mathcal{B}$ and $a \in A - B$,
then there exists $b \in B - A$
such that $(A - a) \cup b \in \mathcal{B}$.



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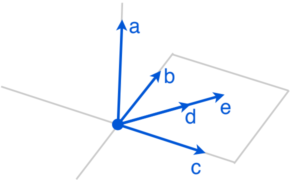

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Definition. (Nakasawa, Whitney, 35)
A set E and a collection $\mathcal{B}$ of subsets of E are a **matroid** if they satisfies properties (B1) and (B2).

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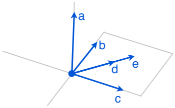
Many matroids in "nature":



1. Linear matroids
E= set of vectors spanning $\mathbb{R}^d$.
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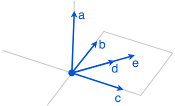

2. Graphical matroids
E= edges of a connected graph G.
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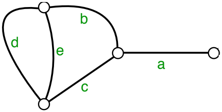

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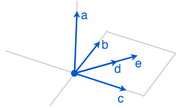

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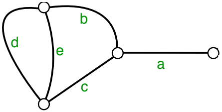

3. Algebraic matroids
E = set of elements in a field extension L/K.
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$a = z^3,\ b = x + y,\ c = x - y$
$d = xy,\ e = x^2y^2.$

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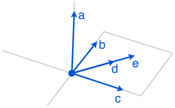

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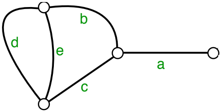

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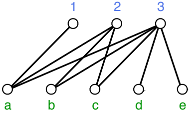
4. Transversal matroids
E = “bottom" vertices of a bipartite graph.
$\mathcal{B}$ = maxl sets that can be matched to the top.

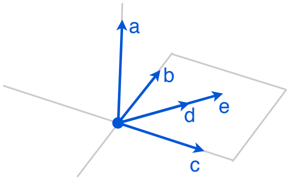


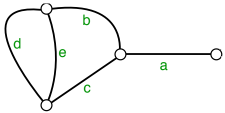
Theorem for matroids $\mapsto$ Theorems for vectors, graphs, field exts, matchings,...

Many points of view.

1. Bases

$\mathcal{B} = \{abc, abd, abe, acd, ace\}$





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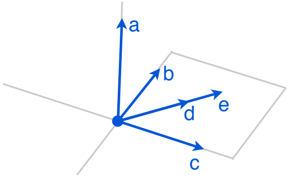

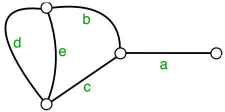

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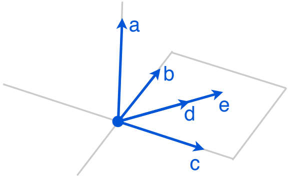
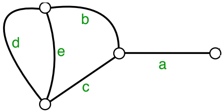
3. Circuits (minimal dependences.)

$\mathcal{C} = \{de, bcd, bce\}$





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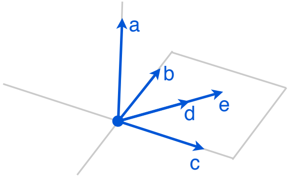

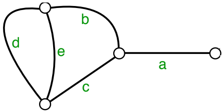

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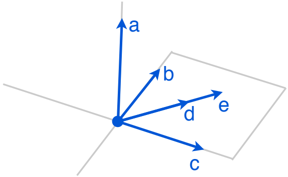

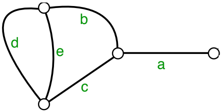

3. Circuits (minimal dependences.)

$\mathcal{C} = \{de, bcd, bce\}$ $\quad \mathcal{BC} = \{d, bc, bc\}$

4. Flats (spanned sets.)

$\mathcal{F} = \{abcde$
$ab, ac, ade, bcde,$
$a, b, c, de,$
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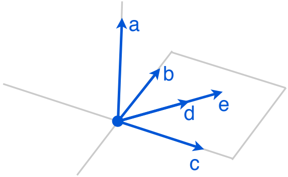
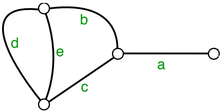
Many points of view.

1. Bases (polytope)
2. Independents (simplicial complex)
3. (Broken) Circuits (monomial ideal)
4. Flats (lattice)





It is as if one were to condense all trends of present day mathematics onto a single finite structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist.

Gian-Carlo Rota

The characteristic polynomial

The **characteristic polynomial** of M is

$$\chi_M(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{r(E)-r(A)}$$

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Monomial ideals:

$$Hilb(\mathbb{R}[x_1, \ldots, x_n]/BC_<(M)) = \left(\frac{t}{t-1}\right)^r \chi_M\left(\frac{t-1}{t}\right)$$

Are matroids geometric?.

(linear matroids) vs. (all matroids):

- Almost any matroid we think of is linear (geometric).
- (Nelson, 18) Almost all matroids are not linear.

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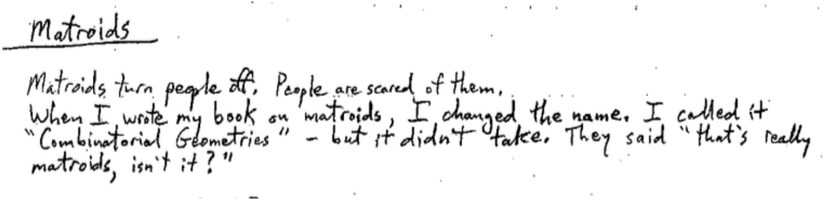
(linear matroids) vs. (all matroids):

- Almost any matroid we think of is linear (geometric).
- (Nelson, 18) Almost all matroids are not linear.

- “Missing axiom" for linear matroids? No. (Mayhew et al, 14)
- This is not a flaw! Matroids are natural geometric objects.

The geometry of matroids.

My main point today.
Matroids are natural geometric objects.


Matroids

Matroids turn people off. People are scared of them.
When I wrote my book on matroids, I changed the name. I called it "Combinatorial Geometries" – but it didn't take. They said "that's really matroids, isn't it?"

Gian-Carlo Rota, Combinatorial Theory, Fall 1998. (Thanks to John Guidi.)

~~The geometry of matroids~~
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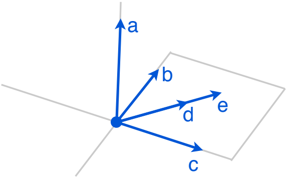
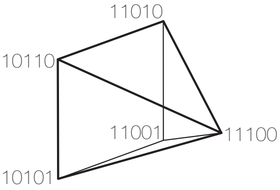
Model 1: Matroid polytopes

Def. (Edmonds 70; Gelfand Goresky MacPherson Serganova 87)
The **matroid polytope** of a matroid M on E is

$$P_M = \mathrm{conv}\{e_B : B \text{ is a basis of } M\} \subset \mathbb{R}^E$$

where e_B is the $0-1$ indicator vector of B.





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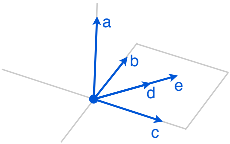

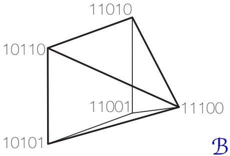

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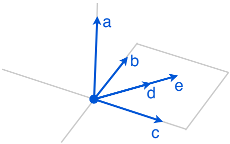

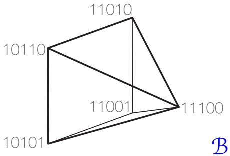

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Matroid polytopes in “nature":

1. Optimization. (Edmonds 70) For a cost function $c : E \to \mathbb{R}$, find the bases $\{b_1, \ldots, b_r\}$ of minimal cost $c(b_1) + \cdots + c(b_r)$.

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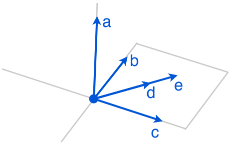

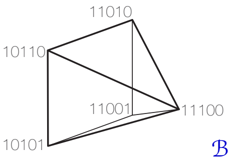

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2. Algebraic geometry. (Gelfand Goresky MacPherson Serganova 87) Understand torus orbits in the Grassmannian.

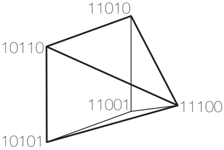
A “Zome tool" characterization of matroids

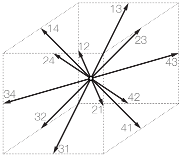
Theorem. (GGMS 87) A collection $\mathcal{B}$ of r-subsets of $[n]$ is a matroid if and only if every edge of the polytope

$$P_M = \text{conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$$

is a translate of vectors $e_i - e_j$ for some i, j.

Def. A **matroid** is a 0-1 polytope with edge directions $e_i - e_j$.





$ij : e_i - e_j$

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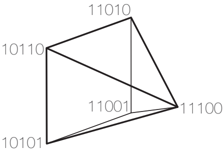

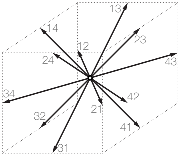

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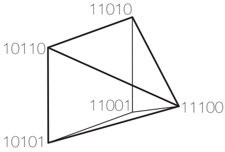
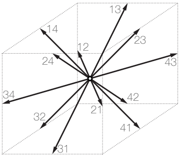
From this geometric viewpoint, all matroids are equally natural.
Matroids provide the correct level of generality!

Applications.



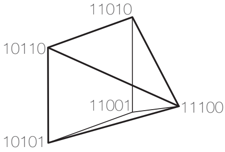


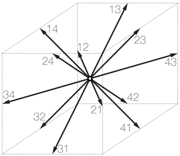
$ij : e_i - e_j$

1. (Laffargue 03) If a matroid polytope cannot be cut into smaller ones, its matroid has finitely many linear $\mathbb{F}$-representations for any fixed $\mathbb{F}$.

$\longmapsto$ theory of matroid subdivisions (Derksen-Fink 10)

Applications.





$ij : e_i - e_j$

1. (Lafforgue 03) If a matroid polytope cannot be cut into smaller ones, its matroid has finitely many linear $\mathbb{F}$-representations for any fixed $\mathbb{F}$.

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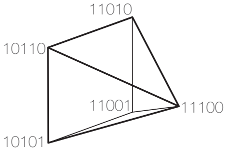

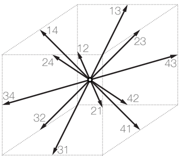

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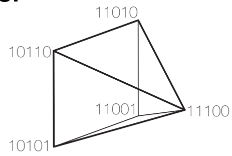

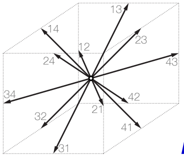

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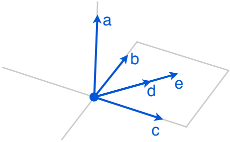
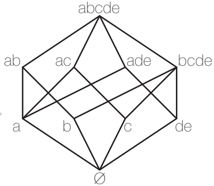
4. $\{e_i - e_j\}$ is the root system for the Lie algebra $\mathfrak{sl}_n$. Other types?

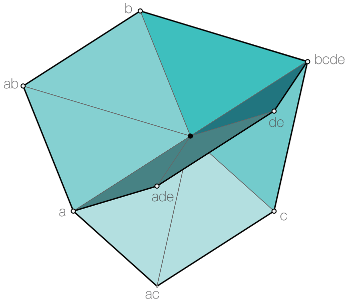
   $\longmapsto$ theory of Coxeter matroids (Gelfand-Serganova 87)

Model 2: Bergman fan

Def/Theorem. (Ardila-Klivans 06)
The *Bergman fan* Σ_M of M is the polyhedral complex with
- rays: $e_F := e_{f_1} + \cdots + e_{f_k}$ for each flat $F = \{f_1, \ldots, f_k\}$
- faces: cone$\{e_F : F \in \mathcal{F}\}$ for each flag $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq E\}$.







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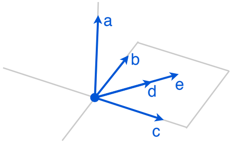

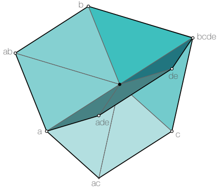

Bergman fans in “nature": Tropical geometry.

algebraic variety $V \mapsto$ Trop(V) polyhedral complex

Trop(V) still knows information about V, and can be studied combinatorially.

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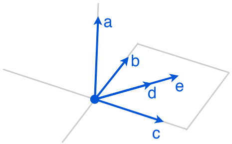

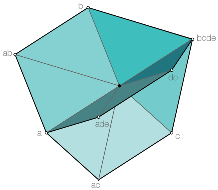

Bergman fans in "nature": Tropical geometry.

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Trop(V) still knows information about V, and can be studied combinatorially.

Question. (Sturmfels 02) Describe Trop(linear space).

Theorem. (Ardila-Klivans 06)
The tropicalization of a linear space $V \subseteq \mathbb{R}^n$ is the Bergman fan $\Sigma_{M(V)}$.

A tropical characterization of matroids

A **tropical variety** is a polyhedral complex “with zero-tension".
It has a **tropical degree**, and AlgDeg(V) = TropDeg(Trop V).

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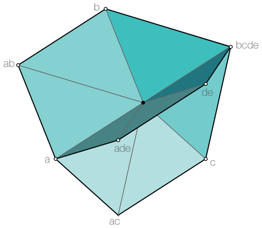
Theorem. (Fink 13) A tropical variety has degree 1 if and only if it is the Bergman fan of a matroid.

Definition. A **matroid** is a tropical variety of degree 1.

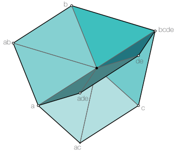


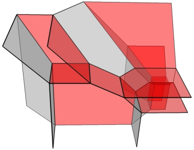
From this geometric viewpoint, all matroids are equally natural.
Matroids provide the correct level of generality!

Applications.

1. A **tropical manifold** is a tropical variety that looks locally like a (Bergman fan of a) matroid.

$\longmapsto$ theory of tropical manifolds (Mikhalkin, Rau, Shaw, ...)





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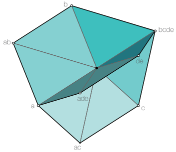

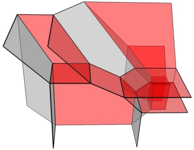

2. (Adiprasito-Huh-Katz 18) A combinatorial **Chow ring** of Σ_M behaves like the cohomology ring of a smooth projective variety. (**!!!**) This gives that the coefficients of the characteristic polynomial

$$\chi_G(q) = w_{v-1}q^{v-1} - w_{v-2}q^{v-2} + \cdots \pm w_1$$

are unimodal and log-concave:

$$w_1 \leq \cdots w_{k-1} \leq w_k \geq w_{k+1} \geq \cdots \geq w_{v-1}$$

$$w_{i-1}w_{i+1} \leq w_i^2 \qquad \text{for } i = 1, \ldots, v-2.$$

This was conjectured by Read (68) and Hoggar (74).

Model 3: conormal fan

Definition. (Ardila-Denham-Huh 17)
A **biflag** of M consists of a flag $\mathcal{F} = \{F_1 \subseteq \cdots \subseteq F_l\}$ of flats and a flag $\mathcal{G} = \{G_1 \supseteq \cdots \supseteq G_l\}$ of coflats (flats of $M^{\perp}$) such that

$$\bigcap_{i=1}^{l} (F_i \cup G_i) = E, \qquad \bigcup_{i=1}^{l} (F_i \cap G_i) \neq E.$$

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All maximal biflags have length $n-2$.

Definition. (Ardila-Denham-Huh 17)
The *conormal fan* $\Sigma_{M,M^\perp}$ is the polyhedral complex in $\mathbb{R}^{E \sqcup E}$ with

- rays $e_F + f_G$ for each flat F and coflat G with $F \cup G = E$
- $cone(\mathcal{F}, \mathcal{G}) := cone\{e_{F_i} + f_{G_i} : 1 \leq i \leq l\}$ for each biflag $(\mathcal{F}, \mathcal{G})$.

Application.

1. The conormal fan seems to be a Lagrangian analog of the Bergman fan. Are conormal fans the tropical Lagrangian linear spaces?

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$$\chi_G(q+1) = h_{v-1}q^{v-1} - h_{v-2}q^{v-2} + \cdots \pm h_1$$

are unimodal, log-concave, and flawless:

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$$h_{i-1}h_{i+1} \leq h_i^2 \qquad \text{for } i = 1, \ldots, v-2.$$

$$h_i \leq h_{s-i} \qquad \text{for the nonzero entries.}$$

This was conjectured by Brylawski (82), Dawson (83) and Swartz (03). It strengthens Adiprasito-Huh-Katz 18 significantly.

muchas gracias.