
linearity
in the
tropics

linearity in the tropics

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connections for women . tropical geometry
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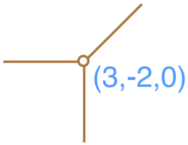

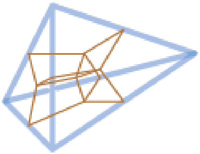

outline

tropical geometry
- tropicalisation
- examples of tropical varieties

tropical linear spaces, part 1: constant coefficients
- linear spaces and matroids
- matroid theory
- the main theorem

tropical linear spaces, part 2: arbitrary coefficients
- from constant coefficients to arbitrary coefficients
- the combinatorics of matroid subdivisions

Summary. Tropical varieties are not simple objects; even tropical linear spaces have a very rich and interesting combinatorial structure which we only partially understand.

Tropical geometry: a general philosophy

Tropicalisation is a very useful general technique:

$$\begin{array}{rcl} \text{algebraic variety} & \mapsto & \text{tropical variety} \\ V & \mapsto & \mathrm{Trop}(V). \end{array}$$

Idea: Obtain information about V from $\mathrm{Trop}(V)$.

o $\mathrm{Trop}(V)$ is simpler, but contains some information about V.

o $\mathrm{Trop}(V)$ is a polyhedral complex, where we can do combinatorics.

Similar to : toric variety $\mapsto$ polyhedral fan

Tropicalisation.

The field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series:

$$f(t) = \alpha_1 t^{r_1} + \alpha_2 t^{r_2} + \cdots, \qquad \alpha_i \in \mathbb{C}, \{r_1 < r_2 < \cdots\} \subset \mathbb{Q}.$$

has valuation deg : $K \to \mathbb{R} \cup \{\infty\} =: \overline{\mathbb{R}}$ where $\deg(f) = r_1$.

Tropicalising points: $\deg : K^n \to \overline{\mathbb{R}}^n$

$$A = (A_1, \ldots, A_n) \mapsto a = (\deg A_1, \ldots, \deg A_n)$$

$$(t^2 + 3t^3 + t^4 + \cdots, t^{1.5} + 2t^2) \mapsto (2, 1.5)$$

Tropicalising polynomials: $\text{Trop} : K[X_1, .., X_n] \to \{f : \mathbb{R}^n \to \mathbb{R}\}$

$A \mapsto \deg A \qquad X + Y \mapsto \min(x, y) \qquad X \cdot Y \mapsto x + y$

$$(t^{1.5} + t^3)X^2 + 2YZ \mapsto \min(1.5 + 2x, y + z)$$

Fundamental Theorem of Tropical Geometry.

Theorem/Defn. (Einsiedler-Lind-Kapranov, Speyer-Sturmfels)
Let I be an ideal in $K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ and let

$$V = V(I) = \{A \in (K^*)^n \mid F(A) = 0 \text{ for } F \in I\}$$

The **tropical variety** $\mathrm{Trop}(V)$ is

$$\begin{aligned} \mathrm{Trop}(V) &:= \{a \in \overline{\mathbb{R}}^n \mid (\mathrm{Trop}\, F)(a) \text{ is achieved twice for } F \in I\} \\ &= \mathrm{cl}\,(\deg A \mid A \in V) \end{aligned}$$

Informally,

$$\begin{aligned} \mathrm{Trop}(V) &:= \text{Solutions of tropical equations} \\ &= \mathrm{cl}\,(\text{Tropicalisation of the solutions}). \end{aligned}$$

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Ex. $V = \{(X, Y, Z) \in (K^*)^3 \mid (t^{-3} + 2)X + (t + 5t^{1.5})Y + Z = 0\}$

1. Tropicalise equations:

$$\mathrm{Trop}\, V = \{(x, y, z) \in \overline{\mathbb{R}}^3 \mid \min(x - 3, y + 1, z) \text{ att. twice}\}.$$

2. Tropicalise solutions:

$$Trop(V) = \mathrm{cl}\,\{(\deg\ X, \deg\ Y, \deg\ Z) \mid (X, Y, Z) \in V\}$$

($2 \subseteq 1$): Exercise.
($1 \subseteq 2$): Harder.

Tropicalisation:

algebraic variety $\mapsto$ tropical variety

$V \mapsto \mathrm{Trop}(V)$.

To apply this technique, we ask two questions:

1. **What does** Trop(V) **know about** V?

Find the right questions in alg. geom. to "tropicalise".

- Gromov-Witten invariants $N^{\mathbb{C}}_{g,d}$ of $\mathbb{CP}^2$ (Mikhalkin)
- Double Hurwitz numbers. (Cavalieri-Johnson-Markwig)

2. **What do we know about** Trop(V)? Not very much!

- (V irred.) Pure, connected in codimension 1. (Bieri-Groves).
- (V Schön) Links have only top homology. (Hacking)

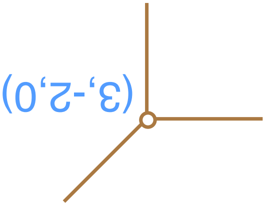
Tropical varieties are 'simpler', not 'simple'. Study them!

Examples of tropical varieties

Example 1. Tropical hyperplanes in $\mathbb{TP}^{n-1}$.

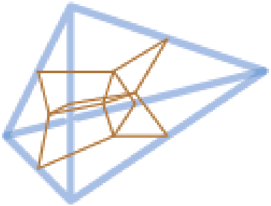
$A_1X_1 + \ldots + A_nX_n = 0 \mapsto \min(x_1 + a_1, \ldots, x_n + a_n)$ ach. twice

$\mathbb{TP}^2$: $\min(x - 3, y + 2, z)$ twice $\mathbb{TP}^3$: $\min(x_1, x_2, x_3, x_4)$ twice



Tropical projective plane $\mathbb{TP}^2$:

$(a, b, c) \sim (a - c, b - c, 0)$

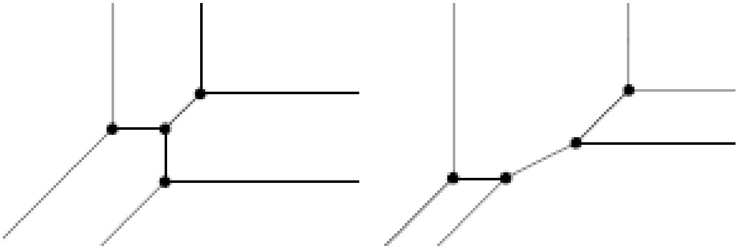


Polar fan of the simplex centered at $-(a_1, \ldots, a_n)$.

Example 2. Tropical conics in $\mathbb{TP}^2$:

$AX^2 + BY^2 + CZ^2 + DXY + EXZ + FYZ = 0 \mapsto$
$\min(a + 2x, b + 2y, \ldots, e + x + z, f + y + z)$ achieved twice.

Two tropical conics:



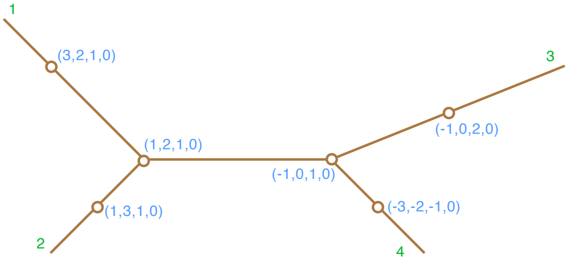
In principle, could have up to $\binom{6}{2} = 15$ edges.
In fact, they all have 4 vertices and 9 edges (3 bounded).

Example 3. A tropical line in $\mathbb{TP}^3$.

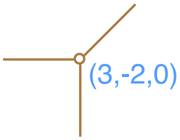
$$L = \text{rowspace} \begin{bmatrix} 1 & t & t^2 & t^3 \\ t^3 & t^2 & t & 1 \end{bmatrix}$$

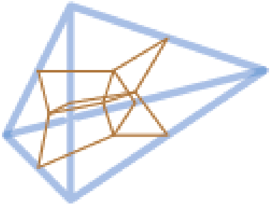
Trop L: The following are attained twice:

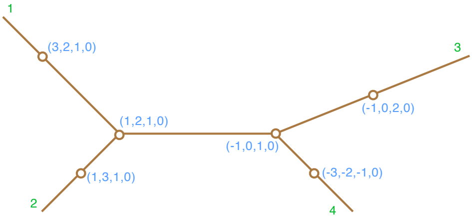
$\min(x_1 + 2, x_2 + 1, x_3 + 2)$, $\min(x_1 + 1, x_2, x_4 + 2)$,
$\min(x_1 + 2, x_3, x_4 + 1)$, $\min(x_2 + 2, x_3 + 1, x_4 + 2)$



The goal of this talk:
To summarize what we know about tropical linear spaces.







Tropical linear spaces, part 1: constant coefficients.

Goal. If V is a linear subspace, describe $\operatorname{Trop} V$.

(Part 1: **Assume that all coefficients are in $\mathbb{C}$.**)

$w \in \operatorname{Trop} V \leftrightarrow$ for each circuit (equation) $a_1 X_{i_1} + \cdots + a_k X_{i_k} = 0$ of V, $\min(w_{i_1}, \ldots, w_{i_k})$ is achieved twice.

Example. $L = \text{rowspace} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$.

$X_1 - X_2 + X_3 = 0\ ,\ X_4 = 2X_3$ Circuits: $123, 34, 124$.

Trop L: $\min(w_1, w_2, w_3), \min(w_1, w_2, w_4), \min(w_3, w_4)$ att. twice.

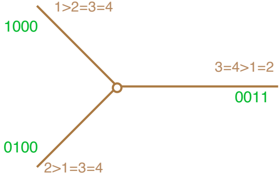
$L = \text{rowspace}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$. Circuits: $123, 34, 124$.

Trop L: $\min(w_1, w_2, w_3), \min(w_1, w_2, w_4), \min(w_3, w_4)$ att. twice.

$w_1 = w_2 < w_5 = w_3 = w_4$ ok $\qquad w_1 = w_3 = w_5 < w_s = w_4$ no

Note.

- w_5 is irrelevant.
- Order of w_1, w_2, w_3, w_4 is either
  - $w_1 > w_2 = w_3 = w_4$,
  - $w_2 > w_1 = w_3 = w_4$, or
  - $w_3 = w_4 > w_1 = w_2$.



So Trop V only depends on the matroid (set of circuits) of V.

For any matroid M (set of circuits) we define

$$\text{Trop}\, M := \{w \in \overline{\mathbb{R}}^E \mid \min_{c \in C} w_c \text{ is achieved twice for all circuits } C.\}$$

(sometimes called the Bergman fan of M.)

This calls for a **crash course in matroid theory.**

Matroid theory, v1: circuits.

Matroid theory: An abstract theory of **independence.**

(Instances: linear, algebraic, graph independence.)

The key properties of (minimal) dependence:

A matroid M on a finite ground set E is a collection $\mathcal{C}$ of circuits (subsets of E) such that:

C0. *$\emptyset$ is not a circuit.*
C1. *No circuit properly contains another.*
C2. *If C_1 and C_2 are circuits and $x \in C_1 \cap C_2$, then $C_1 \cup C_2 - x$ contains a circuit.*

Ex: The matroid of a vector space / config. $L = \text{row}(E)$

(circuits) $\leftrightarrow$ (minl eqns. of L) $\leftrightarrow$ (minl linear deps on cols of E)

Why matroids?

- They are general, applicable, and well-developed.
  Example: Every matroid has a well-defined rank function.
    - Dimension of vector spaces
    - Transcendence degree of a field extension
    - The spanning trees of a graph have the same size.
- Many different (but equivalent) points of view.
    - Matroid polytopes. We need it.
    - Lattice of flats. We need it.
    - Optimization (greedy algorithms). We need it.
- (Our main reason today.) Loosely speaking:

algebraic geometry $\mapsto$ tropical geometry
specialises to
linear algebra $\mapsto$ matroid theory.

Matroid theory, v2: lattices of flats.

E: set of vectors

- flat: (the vectors of E in) span(A) for $A \subseteq E$.
- lattice of flats L_M: the poset of flats ordered by containment.
- order complex $\Delta(\overline{L}_M)$: the simplicial complex of chains of $\overline{L}_M$.

(vertices = flats, faces = flags; $\overline{L}_M = L_M - \{\widehat{0}, \widehat{1}\}$).

$L = \text{rowspace} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}, \mathcal{C} = \{123, 124, 34\}$.

- Flats: $\mathcal{F} = \{\emptyset, 1, 2, 34, 5, 1234, 15, 25, 345, 12345\}$.

Theorem. (Björner, 1980) $\Delta(\overline{L}_M)$ is a pure, shellable simplicial complex. It has the homotopy type of a wedge of $|\mu(L_M)|$ $(r-2)$-dimensional spheres.

The main theorem.

Let $\text{Trop}'M = \text{Trop}M \cap$ (unit sphere).

Theorem. (.f. - Klivans)
$\text{Trop}'(M)$ " = " $\Delta(\bar{L}_M)$.

More precisely, $\Delta(\bar{L}_M)$ is a subdivision of $\text{Trop}'(M)$.

Corollary. (.f. - Klivans) **In constant coefficients**, tropical linear spaces are cones over wedges of $(r-2)$-spheres. The number of spheres is computable combinatorially.

Key observation:

$w_{a_1} = \cdots = w_{a_k} > w_{b_1} = \cdots = w_{b_l} > \cdots$ is in $\text{Trop}(M)$
if and only if $A, A \cup B, A \cup B \cup C, \ldots$ are flats of M.

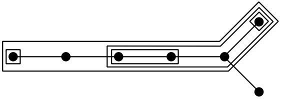
Some interesting special cases.

1. $A_{n-1} = \{e_i - e_j \mid 1 \leq i < j \leq n\}$

• Trop A_{n-1} is the space of phylogenetic trees T_n. (.f. - Klivans)
(T_n also appears naturally in homotopy theory and in $\overline{M}_{0,n}$.)
• T_n has homotopy type $\bigvee_{(n-1)!} S^{n-3}$. (Vogtmann)
• (Chepoi-F. tree reconstruction alg.) = (tropical projection) (.f.)

2. Φ = root system of a finite Coxeter system (W, S)

• Trop′ Φ = (nested set complex of Φ), which encodes De Concini and Procesi's "wonderful compactification" of $\mathbb{C}^n - \mathcal{A}_\Phi$.
• Trop Φ can be described combinatorially as a space of "phylogenetic trees of type W", which come from tubings of the Dynkin diagram. (.f. - Reiner - Williams)



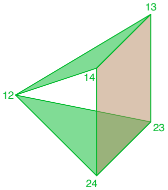
Matroid theory, v3: matroid polytopes

A **basis** of M is a maxl. indept. set. The **matroid polytope** is

$$P_M = \text{conv}(e_{b_1} + \cdots + e_{b_r} \mid \{b_1, \ldots, b_r\} \text{ is a basis.})$$

$L = \text{rowspace} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}, \mathcal{C} = \{123, 124, 34\}$.

- Bases: $\mathcal{B} = \{125, 135, 145, 235, 245\}$
- $P_M = \text{conv}(11001, 10101, 10011, 01101, 01011)$.



Interpretations:

o linear programming and greedy algorithms

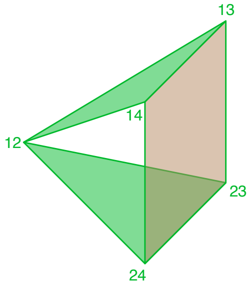
o moment polytope of the closure of a torus orbit in $\text{Gr}(d, n)$

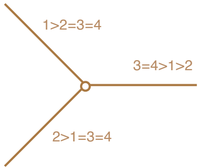
Theorem. (GGMS) A 0-1 polytope is a matroid polytope if and only if all its edges are of the form $e_i - e_j$.

A matroid is loopless if every element is in some basis.

Proposition. (Sturmfels)
Trop M is the fan dual to the loopless faces of P_M:

$$\text{Trop } M = \{ w \in \overline{\mathbb{R}}^E \mid \text{The } w\text{-max face of } P_M \text{ is loopless.} \}$$





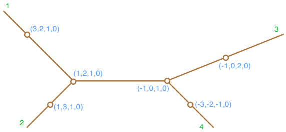
Tropical linear spaces, part 2: arbitrary coefficients.

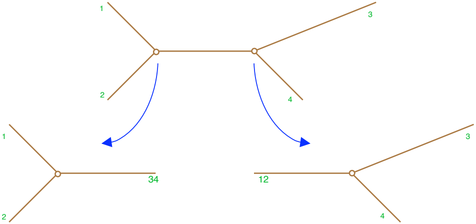
(**from constant to arbitrary coeffs**) Let L be a linear space with arbitrary coeffs and $u \in \operatorname{Trop} L$. The local cone at u is

$$\operatorname{cone}_u \operatorname{Trop} L = \operatorname{Trop} L_u$$

for a linear space L_u *with constant coefficients*.

$L = \text{rowspace} \begin{bmatrix} 1 & t & t^2 & t^3 \\ t^3 & t^2 & t & 1 \end{bmatrix}$, $\operatorname{Trop} L =$

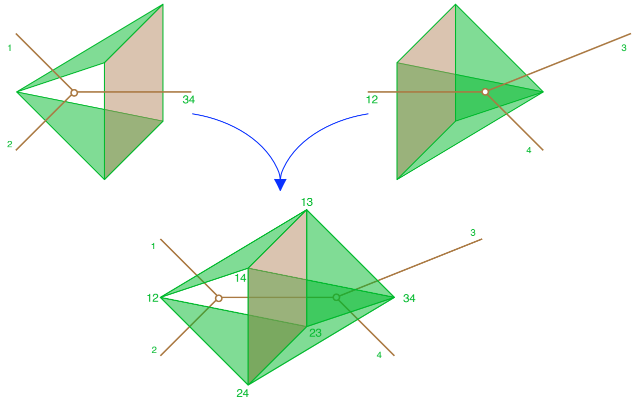




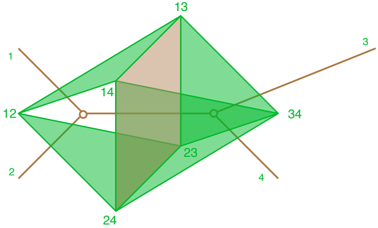
Each local cone is dual to (loopless part of) a matroid polytope.
The matroid polytopes give a subdivision of the hypersimplex

$$\Delta(n,d) = \mathrm{conv}(e_{i_1} + \cdots + e_{i_d} \mid \{i_1, \ldots, i_d\} \subseteq [n])$$

(which is the matroid polytope of a generic vector space.)

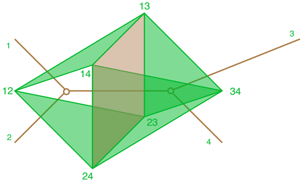


Theorem. (Speyer) A d-dimensional tropical linear space in n-space is dual to a matroid subdivision: a subdivision of $\Delta(n, d)$ into matroid polytopes.



Tropical linear spaces:
constant coeffs. $\mapsto$ matroids arbitrary coeffs. $\mapsto$ matroid subdivisions

Tropical linear spaces:
constant $\mapsto$ matroids
arbitrary $\mapsto$ matroid subdivs.



Other occurrences of matroid subdivisions:

- Kapranov's generalized Lie complexes.

  Chow quot. $Gr(d,n)//\mathbb{T}$ - limits of torus orbit closures in $\mathrm{Gr}(d,n)$
- Hacking, Keel, and Tevelev's very stable pairs.

  generalized hyperplane arrangements.
- Lafforgue's compactif of fine Schubert cells in Grassmannian.

Lafforgue: P_M indecomposable $\rightarrow$ M has finitely many realizations.

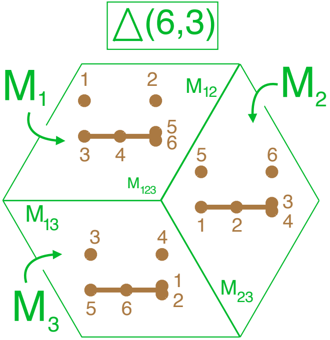
Mnëv: Realization spaces of Ms can have arbitrarily bad singularities.

Matroid subdivisions

How can a matroid polytope can be divided into smaller matroid polytopes?

(Construct? Verify? Prove impossibility?)

One approach:
Find "measures" of a matroid *M* that behave like valuations on P_M.



A function f : Matroids $\to G$ is a matroid valuation if for any subdivision of P_M into $P_{M_1}, \ldots, P_{M_m}$ we have

$$f(M) = \sum_{i=1}^{m} (-1)^{\dim P_M - \dim P_{M_i}} f(M_i) \quad (1)$$

Some matroid valuations:

- $\mathrm{Vol}(P_M)$ (.f.-Benedetti-Doker) (Lam-Postnikov, Stanley)
- $|P_M \cap \mathbb{Z}^n| =$ number of bases of M
- Ehrhart polynomial $E_{P_M}(t) = |tP_M \cap \mathbb{Z}^n|$. (.f. - Doker)
- Tutte polynomial $T_M(x, y)$ (Speyer)
  (the mother of all (del.-contr.) matroid invariants)
- Quasisym function $Q_M(x_1, \ldots, x_n)$ (Billera-Jia-Reiner)
- Invariants coming from K-theory of $\mathrm{Gr}(d, n)$ (Speyer)

Theorem. (Speyer) A d-dimensional tropical linear space in n-space has $\leq \binom{n-i-1}{d-i}\binom{2n-d-1}{i-1}$ i-dimensional faces.

He uses a mysterious invariant $g_M(t)$ from K-theory. What does it mean combinatorially? If we knew, we could prove:

Conjecture. This bound holds for any matroid subdivision.

A very general matroid valuation.

Define V : Matroids $\to G$ by:

$$V(M) = \sum_{\pi \in S_n} (\pi, r(\pi_1), r(\pi_1, \pi_2), \ldots, r(\pi_1, \ldots, \pi_n))$$

where G is the free abelian group generated by such symbols.

For $L = \text{rowspace} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$,

$$V(M) = (1234, 1, 2, 2, 2) + \cdots + (3421, 1, 1, 2, 2) + \cdots.$$

Theorem. (.f. - Fink - Rincón, Derksen)
V is a matroid valuation.

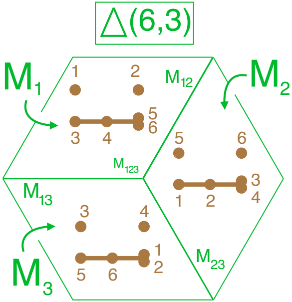
$$V(M) = \sum_{\pi \in S_n} (\pi, (r(\pi_1), r(\pi_1, \pi_2), \ldots, r(\pi_1, \ldots, \pi_n))$$

Theorem. (.f. - Fink - Rincón, Derksen)
V is a matroid valuation.

Example. For the subdivision of $\Delta(6,3)$

$V(M) = V(M_1) + V(M_2) + V(M_3)$

$-V(M_{12}) - V(M_{13}) - V(M_{23}) + V(M_{123})$



The summands with $\pi = 132456$ give

(writing (132456, 1, 2, 3, 3, 3, 3) $\rightarrow$ (1, 2, 3, 3))

$(1,2,3,3) = (1,2,3,3) + (1,2,2,3) + (1,2,2,2)$

$-(1,2,2,3) - (1,2,2,2) - (1,2,2,2) + (1,2,2,2)$

Idea of proof. Interpret each term like

$$(1,2,2,2) - (1,2,2,2) - (1,2,2,2) + (1,2,2,2) = 0$$

as a reduced Euler characteristic of a contractible space.

All matroid valuations.

$$V(M) = \sum_{\pi \in S_n} (\pi, (r(\pi_1), r(\pi_1, \pi_2), \ldots, r(\pi_1, \ldots, \pi_n))$$

Theorem. (Derksen - Fink)
V is a **universal** matroid valuation.

Theorem. (Derksen - Fink)
Let $v(n, r)$ be the rank of the abelian group of valuations on matroids of n elements and rank r. Then

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} v(n, r) \frac{x^{n-r} y^r}{n!} = \frac{x - y}{xe^{-x} - ye^{-y}}.$$

So in principle we know how far we can push this approach.
In practice there is more to do.

summary

- We do not understand tropical varieties very well yet.
- We understand tropical linear spaces to some extent.
  - Locally, they “are" matroids.
  - Globally, they “are" matroid subdivisions.
  - We know many things about matroids, and a few things about matroid subdivisions.

some future directions

- Understand matroid subdivisions better. Systematic construction? Mixed subdivisions? Secondary polytope?
- Generalize this story to subdivisions of Coxeter matroids and tropical homogeneous spaces (under certain hypotheses, to be determined). (.f. - Rincón - Velasco)
- What about general tropical varieties?

many thanks !!!

Papers available at:

http://math.sfsu.edu/federico

http://front.math.ucdavis.edu