Tutte Polynomials of Hyperplane Arrangements

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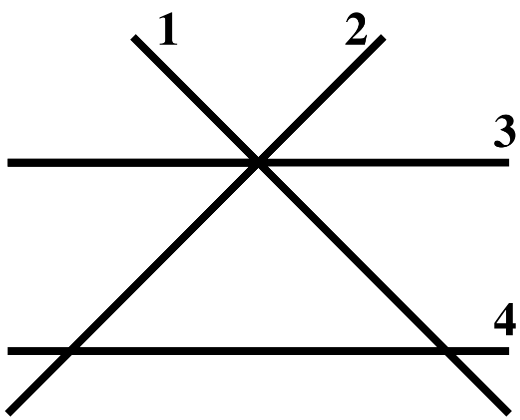
The Tutte polynomial

Let $\mathbb{k}$ be a field. A **hyperplane arrangement** $\mathcal{A}$ is a set of affine hyperplanes in $\mathbb{k}^n$.

- A subset $\mathcal{B} \subseteq \mathcal{A}$ is **central** if $\cap\mathcal{B} \neq \emptyset$.
- The **rank of a central subset** $\mathcal{B}$ is $r(\mathcal{B}) = n - \dim \cap\mathcal{B}$.
- The **rank of the arrangement** $\mathcal{A}$ is the largest rank of a central subset of $\mathcal{A}$, and it is denoted r.

The **Tutte polynomial** of $\mathcal{A}$ is

$$T_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \text{central}}} (q-1)^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|-r(\mathcal{B})}.$$



central subset	rank	contribution
$\emptyset$	0	$(q-1)^2(t-1)^0$
$1,2,3,4$	1	$(q-1)^1(t-1)^0$
$12,13,14,23,24$	2	$(q-1)^0(t-1)^0$
123	2	$(q-1)^0(t-1)^1$

$$\begin{aligned} T(q,t) &= (q-1)^2 + 4(q-1) + 5 + (t-1) \\ &= q^2 + 2q + t + 1 \end{aligned}$$

Why is the Tutte polynomial interesting?

The Tutte polynomial has been defined for graphs [Tutte, 1947] and for matroids [Crapo, 1969].

For graphs:

- $T(1,1)$ = number of spanning trees.
- $T(2,0)$ = number of acyclic orientations of the edges. [Stanley, 1973]
- $T(0,2)$ = number of totally cyclic orientations of the edges. [Stanley, 1980]
- $(-1)^{v-c}\, q^c\, T(1-q,0)$ = number of proper q-colorings of the vertices. [Tutte, 1947]
- $(-1)^{e-v+c}\, T(0,1-t)$ = number of nowhere zero t-flows, under any orientation of the edges. [Crapo, 1969]

For matroids:

- $T(1,1)$ = number of bases.
- $T(2,1)$ = number of independent sets.
- $T(1,2)$ = number of spanning sets.
- $(-1)^{r-k}\left[q^k\right] T(1-q,0)$ = number of k-subsets with no broken circuits. [Whitney, 1935]
- For a matroid represented by the columns of a matrix M over $\mathbb{Q}$, p and q large enough prime powers and $a+b=1$,

$$a^{r(M^*)}b^{r(M)}T_M\left(\frac{1+(p-1)a}{b}, \frac{1+(q-1)b}{a}\right) = \sum a^{|s(x)|}b^{|s(y)|}$$

where the sum is over all $(x,y) \in \text{row}(M) \times \ker(M) \subseteq \mathbb{F}_p^n \times \mathbb{F}_q^n$ such that $s(x) \cap s(y) = \emptyset$. [Reiner, 1997]

The **characteristic polynomial** $\chi_{\mathcal{A}}(q)$ of a hyperplane arrangement is also interesting.

- $(-1)^n\chi(-1)$ = number of regions of $\mathbb{R}^n - \bigcup \mathcal{A}_{\mathbb{R}}$. [Zaslavsky, 1975]
- $(-1)^n\chi(1)$ = number of bounded regions of $\mathbb{R}^n - \bigcup \mathcal{A}_{\mathbb{R}}$. [Zaslavsky, 1975]
- $(-q)^n\chi(\frac{-1}{q})$ = Poincaré polynomial of $\mathbb{C}^n - \bigcup \mathcal{A}_{\mathbb{C}}$. [Orlik and Solomon, 1980]
- $(-q)^n\chi(\frac{-1}{q})$ = Poincaré polynomial of the Orlik - Solomon algebra of $\mathcal{A}$. [Orlik and Solomon, 1980]
- With our definition of $T(q,t)$, $\chi(q) = (-1)^r q^{n-r} T(1-q, 0)$. [Whitney, 1935] [Postnikov, 1997]

What can we say about $T_{\mathcal{A}}(q,t)$?

A central arrangement $\mathcal{A}$ has a matroid $M_{\mathcal{A}}$ associated to it. $M_{\mathcal{A}}$ is determined by the vectors normal to the hyperplanes of $\mathcal{A}$.

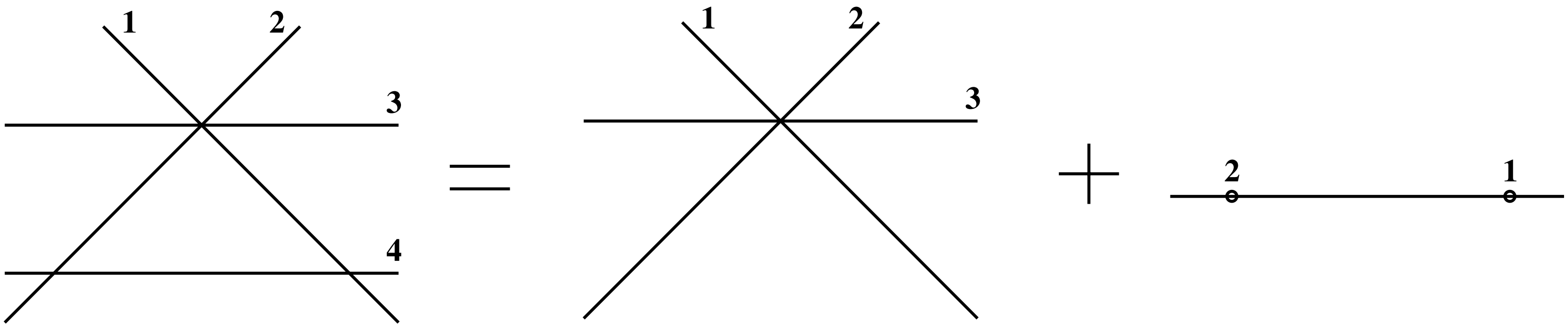
Since $T_{\mathcal{A}}(q,t) = T_{M_{\mathcal{A}}}(q,t)$, central arrangements inherit all the Tutte polynomial properties of a matroid in this way.

For affine arrangements some of these properties extend easily, others extend with more difficulty, and others do not seem to extend.

Deletion - contraction.

If the hyperplane H is neither a loop nor a coloop, then

$$T_{\mathcal{A}}(q,t) = T_{\mathcal{A}-H}(q,t) + T_{\mathcal{A}/H}(q,t).$$



Furthermore, any function satisfying such a recurrence is an evaluation of the Tutte polynomial. These functions are called **Tutte - Grothendieck invariants.**

Examples: number of central subsets, number of regions.

$T(q,t)$ has non-negative coefficients.

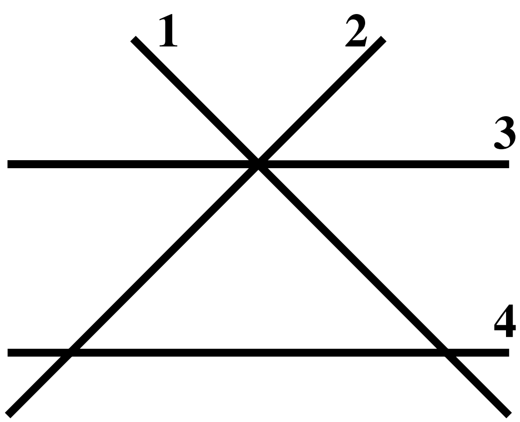
A set $B \subseteq \mathcal{A}$ is a **basis** if it is central and $r(B) = |B| = r$.

Number the hyperplanes of $\mathcal{A}$. For a basis B:

- Say that $e \notin B$ is an **external activity element for** B if $B \cup e$ is central and $r(B_{>e} \cup e) = r(B_{>e})$. $E(B)$ is the set and $e(B)$ is the number of external activity elements.
- Say that $i \in B$ is an **internal activity element for** B if $r((B - i) \cup \mathcal{A}_{<i}) < r$. $I(B)$ is the set and $i(B)$ is the number of internal activity elements.

Theorem.

$$T(q,t) = \sum_{B \text{ basis}} q^{i(B)} t^{e(B)}$$



B	$I(B)$	$E(B)$	$B - I \cup E$
12	12	-	$\emptyset, 1, 2, 12$
13	1	-	$3, 13$
14	1	-	$4, 14$
23	-	1	$23, 123$
24	-	-	24

$$T(q,t) = q^2 + 2q + t + 1$$

Key fact:

Each central subset $\mathcal{B} \subseteq \mathcal{A}$ can be written uniquely as $\mathcal{B} = B - I \cup E$ where B is a basis, $I \subseteq I(B)$ and $E \subseteq E(B)$. Furthermore, its rank is $r(\mathcal{B}) = r - |I|$.

$$T_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (q-1)^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|-r(\mathcal{B})}$$

$$= \sum_{B} \sum_{I \subseteq I(B)} \sum_{E \subseteq E(B)} (q-1)^{r-r(B-I \cup E)}(t-1)^{|B-I \cup E|-r(B-I \cup E)}$$

$$= \sum_{B} \sum_{I \subseteq I(B)} \sum_{E \subseteq E(B)} (q-1)^{|I|}(t-1)^{|E|}$$

$$= \sum_{B} (1+(q-1))^{|I(B)|}(1+(t-1))^{|E(B)|}$$

$$= \sum_{B} q^{i(B)} t^{e(B)}.$$

A finite field method.

Let $\overline{\chi}(q,t) = (t-1)^r \, T\left(\frac{q+t-1}{t-1}, t\right)$.

Theorem. Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement in $\mathbb{R}^n$. Let q be a large enough prime power, and let $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then

$$q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}$$

where $h(p)$ denotes the number of hyperplanes of $\mathcal{A}_q$ that p lies on.

The same result holds for subspace arrangements.

An example.

$\mathcal{H}_n$: $x_i = 0$ $(1 \le i \le n)$.

$$\begin{aligned}
\overline{\chi}_{\mathcal{H}_n}(q,t) &= \sum_{p \in \mathbb{F}_q^n} t^{h(p)} \\
&= \sum_{k=0}^{n} \left(\binom{n}{k} (q-1)^{n-k} \right) t^k \\
&= (q+t-1)^n.
\end{aligned}$$

Applications and specializations.

- Summing over all $x \leq y$ in the intersecton poset $L_{\mathcal{A}}$,
  $\overline{\chi}_{\mathcal{A}}(q,t) = \sum \mu(x,y) q^{n-r(y)} t^{h(x)}$.
- Pick a random subarrangement $\mathcal{B} \subseteq \mathcal{A}$, by independently deleting each hyperplane from $\mathcal{A}$ with probability t. Then
  $E[\chi_{\mathcal{B}}(q)] = q^{n-r} \overline{\chi}_{\mathcal{A}}(q,t)$.
- [Greene, 1976] Let C be an r-dimensional subspace of $\mathbb{F}_q^n$. Let U be a matrix whose rows form a basis for C, and let $M(U)$ be the matroid determined by the columns of U. For each $v \in C$, let $w(v)$ be the number of nonzero coordinates of v. Then
  $\sum_{v \in C} t^{w(v)} = t^n \overline{\chi}_{M(U)} \left(q, \frac{1}{t}\right)$.

Coboundary polynomials of specific arrangements.

$\mathcal{A}_n$: $x_i = x_j \ (1 \leq i < j \leq n)$.

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\mathcal{A}_n}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!} \right)^q$$

$\mathcal{B}_n$: $x_i = x_j,\ x_i + x_j = 0 \ (1 \leq i < j \leq n),\ x_i = 0 \ (1 \leq i \leq n)$.

$$\sum_{n \geq 0} \overline{\chi}_{\mathcal{B}_n}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!} \right)^{\frac{q-1}{2}} \left(\sum_{n \geq 0} t^{n^2} \frac{x^n}{n!} \right)$$

$\mathcal{D}_n$: $x_i = x_j$, $x_i + x_j = 0$ $(1 \leq i < j \leq n)$.

$$\sum_{n \geq 0} \overline{\chi}_{\mathcal{D}_n}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!} \right)^{\frac{q-1}{2}} \left(\sum_{n \geq 0} t^{\,n(n-1)} \frac{x^n}{n!} \right)^{q}$$

$\mathcal{A}_n^{\#}$: $x_i - x_j = a_{ij}$ $(1 \leq i < j \leq n)$, where the a_{ij}'s are generic.

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\mathcal{A}_n^{\#}}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} f(n) \frac{x^n (q-1)^n}{n!} \right)^{\frac{t}{q-1}}$$

where $f(n)$ denotes the number of forests on $[n]$.

The threshold arrangement.

$\mathcal{T}_n$: $x_i + x_j = 0$ $(1 \leq i < j \leq n)$.

$$\overline{\chi}_{\mathcal{T}_n}(q,t) = \sum_G q^{bc(G)}(t-1)^{e(G)},$$

summing over all graphs G on $[n]$. Here $bc(G)$ is the number of connected components of G which are bipartite, and $e(G)$ is the number of edges of G. Also,

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{T}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} t^{\binom{n}{2}}\frac{x^n}{n!}\right)\left(\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}t^{k(n-k)}\frac{x^n}{n!}\right)^{\frac{q-1}{2}}.$$

The Linial arrangement.

$\mathcal{L}_n$: $x_i - x_j = 1$ $(1 \leq i < j \leq n)$.

$$q\,\overline{\chi}_{\mathcal{L}_n}(q,t) = \sum_P q^{c(P)}(t-1)^{e(P)}$$

summing over all naturally labeled graded posets P on $[n]$. Here $c(P)$ is the number of connected components of the Hasse diagram of P, and $e(P)$ is the number of edges.

Let $A_k(x, t-1) = \sum t^{\mathrm{id}(w)} \frac{x^{|w|}}{|w|!}$, summing over all words w of letters chosen from $[k]$. Here $\mathrm{id}(w)$ is the total number of times that some letter $i+1$ occurs before some letter i in w. Then

$$1 + q\sum_{n\geq 1} \overline{\chi}_{\mathcal{L}_n}(q,t)\frac{x^n}{n!} = \left(\lim_{k\to\infty} \frac{A_k}{A_{k-1}}\right)^q.$$

The number l_n of regions of the Linial arrangement $\mathcal{L}_n$ is equal to the number of **alternating trees** on $[n+1]$. These are trees such that each vertex is either smaller or larger than all of its neighbors. [Athanasiadis, 1996][Postnikov, 1997]

We get a new formula for the number of alternating trees. Let

$$\frac{1 + ye^{x(1+y)}}{1 - y^2 e^{x(1+y)}} = \sum_{n \geq 0} L_n(x) y^n.$$

Then

$$\sum_{n \geq 0} (-1)^n l_n \frac{x^n}{n!} = \lim_{k \to \infty} \frac{L_{k-1}(x)}{L_k(x)}.$$

The Shi arrangement.

$\mathcal{S}_n$: $x_i - x_j = 0, 1 \ (1 \leq i < j \leq n)$.

The number of regions is $(n+1)^{n-1}$. [Shi, 1986]

$$\sum_{n \geq 0} (-1)^n (n+1)^{n-1} \frac{x^n}{n!} = \lim_{k \to \infty} \frac{\sum_n (k-1-n)^n \frac{x^n}{n!}}{\sum_n (k-n)^n \frac{x^n}{n!}}.$$

The Catalan arrangement.

$\mathcal{C}_n$: $x_i - x_j = -1, 0, 1 \ (1 \leq i < j \leq n)$.

The number of regions is $n! \, C_n$. [Stanley, 1996]

$$\sum_{n \geq 0} (-1)^n C_n x^n = \lim_{k \to \infty} \frac{\sum_n \binom{k-1-n}{n} x^n}{\sum_n \binom{k-n}{n} x^n}.$$