The closure of a linear space in a product of lines

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Joint work with:

  

Adam Boocher (UC Berkeley → U Edinburgh)

The closure of a linear space in a product of lines.

Federico Castillo (UC Davis), José Samper (U Washington)

The topology of the external activity complex of a matroid.

Summary.

• We compute several (geom./alg.) invariants of a (variety/ideal) in terms of the combinatorial invariants of a matroid.

• The initial ideals give rise to beautiful simplicial complexes associated to a matroid. They are shellable spheres or balls.

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GEOMETRY

CLOSURES OF LINEAR SPACES. Let $L \subset \mathbb{C}^n$ be a **linear space**.

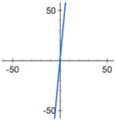
Embedding each line $\mathbb{C}$ into the **projective line** $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
We get an embedding

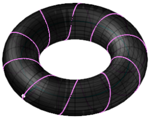
$$\mathbb{C}^n \hookrightarrow (\mathbb{P}^1)^n.$$

which leads us to the **closure** $\widetilde{L}$ of the linear space L in $(\mathbb{P}^1)^n$.

$$L \hookrightarrow \widetilde{L}$$

(Real) cartoon:

$L =$ 

$\widetilde{L}$ “=" 

Our geometric goal.
Study the variety $\widetilde{L}$.

ALGEBRA

HOMOGENIZATION. To **homogenize** a polynomial

$$f(x_1, \ldots, x_n) \quad \mapsto \quad \widetilde{f}(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

we substitute $f(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n})$ and clear denominators.
The homogenization of an ideal I is $\widetilde{I} = \{\widetilde{f} \,:\, f \in I\}$.

Consider a **linear ideal**, such as

$$I = \langle x_1 + x_2 + x_6,\ x_2 - x_3 + x_5,\ x_3 + x_4 \rangle$$

Its homogenization is

$$\widetilde{I} = \langle x_1 y_2 y_6 + y_1 x_2 y_6 + y_1 y_2 x_6,\ x_2 y_3 y_5 - y_2 x_3 y_5 + y_2 y_3 x_5, \ldots \rangle .$$

Our algebraic goal.
Study the homogenization of a linear ideal.

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Our algebraic goal.
Study the homogenization of a linear ideal.

Suppose I is a linear ideal, such as

$$I = \langle x_1 + x_2 + x_6,\ x_2 - x_3 + x_5,\ x_3 + x_4 \rangle .$$

Then we homogenize every polynomial in I:

$$\widetilde{I} = \langle x_1 y_2 y_6 + y_1 x_2 y_6 + y_1 y_2 x_6,\ x_2 y_3 y_5 - y_2 x_3 y_5 + y_2 y_3 x_5, \ldots \rangle .$$

BUT we can't just homogenize the generators of I:

$$\widetilde{I} \neq \langle x_1 y_2 y_6 + y_1 x_2 y_6 + y_1 y_2 x_6,\ x_2 y_3 y_5 - y_2 x_3 y_5 + y_2 y_3 x_5,\ x_3 y_4 + y_3 x_4 \rangle$$

Ex: $f = x_2 + x_4 + x_5 \in I$, but $\widetilde{f} = x_2 y_4 y_5 + y_2 x_4 y_5 + y_2 y_4 x_5$ is missing.

It is generally difficult to write down explicit equations for ideals.

Motivating Question.
What is the best generating set for $\widetilde{I}$?

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COMBINATORICS

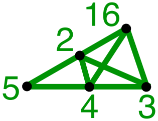
MATROIDS. Suppose

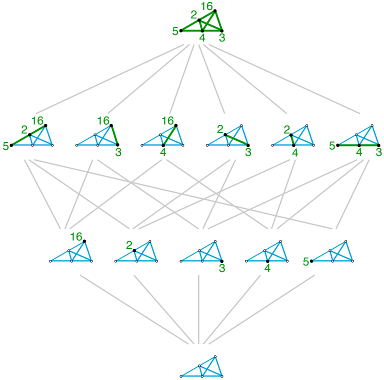
$$I = \langle x_1 + x_2 + x_6,\ \ x_2 - x_3 + x_5,\ \ x_3 + x_4 \rangle \subset \mathbb{C}[x_1, \ldots, x_6]$$

Use the gens of I as rows of a matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Treat the 6 columns as points.





The **matroid** M is the list :

Flats: • points 16, 2, 3, 4, 5
• lines 1256,136,146, 23, 24, 345

Bases: 123, 124, 134, 135, 145, 234, 235, 236, 245, 246, 346, 356, 456

Key. Use the matroid (comb.) to study the variety (geom.) and ideal (alg.)

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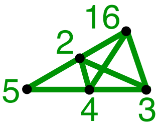

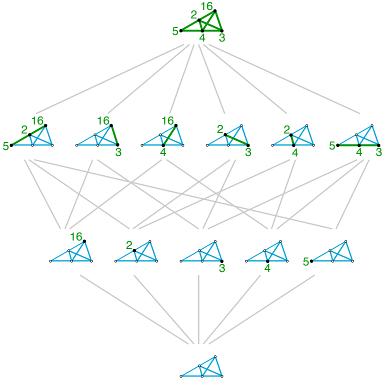

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COMBINATORICS → GEOMETRY

DEGREE. The **degree** of a d-dim projective variety V is:

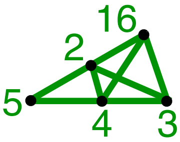
$$\deg V = |V \cap P|$$

the number of intersection points with a codimension d-plane P.

Ex. V = 2-sphere in $\mathbb{C}^3$ $\longrightarrow$ $\deg V = 2$.
Ex. V = linear d-space $\longrightarrow$ $\deg V = 1$.

Question. What is the **degree** of our variety $\widetilde{L}$?

Theorem. (A.– Boocher '13) The degree of $\widetilde{L}$ equals the number of **bases** of the matroid M.



In our example we have 13 bases (triples of points not on a line)
123, 124, 134, 135, 145, 234, 235, 236, 245, 246, 346, 356, 456.

So $\deg \widetilde{L} = 13$.

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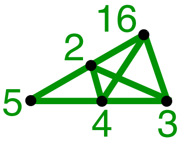

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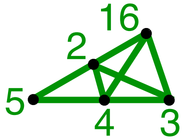
COMBINATORICS ⟶ ALGEBRA

GENERATORS. Warmup: Generators of linear ideals.

TAKE 1. We have



$$I = \langle x_1 + x_2 + x_6,\ x_2 - x_3 + x_5,\ x_3 + x_4 \rangle$$

First observation:

- support of generators of I: $126, 235, 34$
- complements: 345, 146, 1256 are lines (hyperplanes!)

Why not include all hyperplanes? Each gives a unique equation in I.

TAKE 2. Use the **cocircuits**:

$$\begin{aligned} I \quad = \quad & \langle x_1 + x_2 + x_6,\ x_1 + x_3 - x_5 + x_6,\ x_1 - x_4 - x_5 + x_6, \\ & x_2 - x_3 + x_5,\ x_2 + x_4 + x_5,\ x_3 + x_4 \rangle \end{aligned}$$

- support of generators of I: $126, 1356, 1456, 235, 245, 34$
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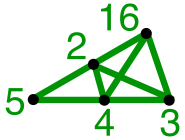

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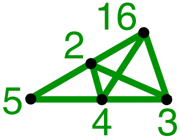
GOOD: This is a **minimal** set of generators.



TAKE 2. We have

$$I = \langle x_1 + x_2 + x_6,\ x_1 + x_3 - x_5 + x_6,\ x_1 - x_4 - x_5 + x_6,\ x_2 - x_3 + x_5,\ x_2 + x_4 + x_5,\ x_3 + x_4 \rangle$$

- support of generators of I: 126, 1356, 1456, 235, 245, 34 (cocircuits)
- complements: 345, 24, 23, 146, 136, 1256 (hyperplanes)

BAD: This is not a **minimal** set of generators.

GOOD: It is the **universal Gröbner basis.** (Great for computations.)

(Sturmfels '96)

GENERATORS. What are the generators of the homogenization $\widetilde{I}$?

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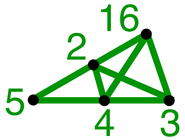
$$\widetilde{I} \neq \langle x_1 y_2 y_6 + y_1 x_2 y_6 + y_1 y_2 x_6, \quad x_2 y_3 y_5 - y_2 x_3 y_5 + y_2 y_3 x_5, \quad x_3 y_4 + y_3 x_4 \rangle$$

ALL BAD: This is not even a set of generators.
(E.g., it doesn't contain $x_2 y_4 y_5 + y_2 x_4 y_5 + y_2 y_4 x_5$.)



TAKE 2. What if we try homogenizing all the cocircuits?

$$\widetilde{I} = \langle x_1 y_2 y_6 + y_1 x_2 y_6 + y_1 y_2 x_6, \quad x_1 y_3 y_5 y_6 + y_1 x_3 y_5 y_6 - y_1 y_3 x_5 y_6 + y_1 y_3 y_5 x_6,$$
$$x_1 y_4 y_5 y_6 - y_1 x_4 y_5 y_6 - y_1 y_4 x_5 y_6 + y_1 y_4 y_5 x_6, \quad x_2 y_3 y_5 - y_2 x_3 y_5 + y_2 y_3 x_5,$$
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Theorem. (A. – Boocher '13) The homogenized cocircuits of I
- are a **minimal** set of generators for $\widetilde{I}$.
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These 2 properties rarely occur together. When they do, $\widetilde{I}$ is **robust**.

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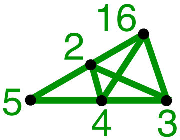

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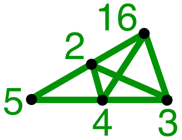
SYZYGIES. We know the generators (complements ↔ hyperplanes).

$$\widetilde{I} = \langle x_1y_2y_6 + y_1x_2y_6 + y_1y_2x_6, \quad x_1y_3y_5y_6 + y_1x_3y_5y_6 - y_1y_3x_5y_6 + y_1y_3y_5x_6, \\ x_1y_4y_5y_6 - y_1x_4y_5y_6 - y_1y_4x_5y_6 + y_1y_4y_5x_6, \quad x_2y_3y_5 - y_2x_3y_5 + y_2y_3x_5, \\ x_2y_4y_5 + y_2x_4y_5 + y_2y_4x_5, \quad x_3y_4 + y_3x_4\rangle .$$



What are the linear relations (**syzygies**) among them? Here are two:

$$y_2y_5(x_3y_4 + y_3x_4) + y_4(x_2y_3y_5 - y_2x_3y_5 + y_2y_3x_5) - y_3(x_2y_4y_5 + y_2x_4y_5 + y_2y_4x_5) = 0$$

$$(x_2y_5+y_2x_5)(x_3y_4+y_3x_4)-x_4(x_2y_3y_5-y_2x_3y_5+y_2y_3x_5)-x_3(x_2y_4y_5+y_2x_4y_5+y_2y_4x_5) = 0$$

But there are seven more! How to describe them? Organize them?

Hint: these two have support 2345 whose complement is 16, a point.

SYZYGIES. We have:

- 6 generators. (complements: 1256, 136, 146, 23, 24, 345)

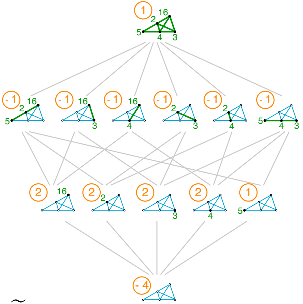
$$\widetilde{I} = \langle x_1y_2y_6 + y_1x_2y_6 + y_1y_2x_6, \quad x_1y_3y_5y_6 + y_1x_3y_5y_6 - y_1y_3x_5y_6 + y_1y_3y_5x_6,$$
$$x_1y_4y_5y_6 - y_1x_4y_5y_6 - y_1y_4x_5y_6 + y_1y_4y_5x_6, \quad x_2y_3y_5 - y_2x_3y_5 + y_2y_3x_5,$$
$$x_2y_4y_5 + y_2x_4y_5 + y_2y_4x_5, \quad x_3y_4 + y_3x_4\rangle .$$

- (2 of) 9 relations (**syzygies**). (compls: 16, 16, 2, 2, 3, 3, 4, 4, 5)

$$y_2y_5(x_3y_4 + y_3x_4) + y_4(x_2y_3y_5 - y_2x_3y_5 + y_2y_3x_5) - y_3(x_2y_4y_5 + y_2x_4y_5 + y_2y_4x_5) = 0$$

$$(x_2y_5 + y_2x_5)(x_3y_4 + y_3x_4) - x_4(x_2y_3y_5 - y_2x_3y_5 + y_2y_3x_5) - x_3(x_2y_4y_5 + y_2x_4y_5 + y_2y_4x_5) = 0.$$

- 4 rels. among the rels. (**2nd syzygies**). (complement: $\emptyset$)



The picture on the right (**lattice of flats**) organizes the syzygies beautifully.

The free resolution of $\widetilde{I}$ as an S-module is:

$$S^4 \to S^9 \to S^6 \to S \to S/\widetilde{I} \to 0.$$

Theorem. (A. – Boocher '13)
The syzygies of $\widetilde{I}$ are supported on $[n] - F$ where F is a **flat** of M. There are $|\mu^*(F)|$ ith syzygies on $[n] - F$, where

$$\mu([n]) = 1, \qquad \mu^*(F) = -\sum_{G \supset F} \mu^*(G)$$

is the **dual Möbius function** of M. Equivalently,

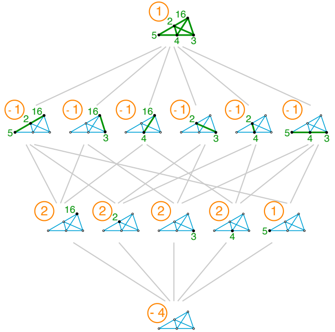
$$\beta_{i,\mathbf{a}}(S/I(\widetilde{L})) = \begin{cases} |\mu^*(F)| & \text{if } \mathbf{a} = e_{[n]-F},\ i = r - r(F) \\ 0 & \text{otherwise.} \end{cases}$$

Hence:

- $\widetilde{I}$ is **Cohen-Macaulay**.
- The Betti numbers of $\widetilde{I}$ are equal to the coefficients of the **cocharacteristic polynomial** of the matroid of I.

$$S^4 \to S^9 \to S^6 \to S \to S/\widetilde{I} \to 0$$

$$\chi^*(q) = -4 + 9q - 6q^2 + q^3$$



TOPOLOGY

A key ingredient is to understand the initial ideals

$$\mathrm{in}_< I(\widetilde{L}) = \langle x_1 y_2 y_6,\ x_1 y_3 y_5 y_6,\ x_1 y_4 y_5 y_6,\ x_2 y_3 y_5,\ x_2 y_4 y_5,\ x_3 y_4 \rangle$$

We have

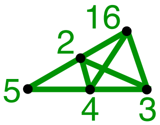
$$\begin{aligned} \mathrm{in}_< I(\widetilde{L}) \quad = \quad & \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_4 \rangle \cap \langle x_1, y_3, y_4 \rangle \cap \langle x_1, x_3, y_5 \rangle \cap \langle x_1, y_4, y_5 \rangle \cap \\ & \langle y_2, y_3, y_4 \rangle \cap \langle y_2, x_3, y_5 \rangle \cap \langle x_2, x_3, y_6 \rangle \cap \langle y_2, y_4, y_5 \rangle \cap \langle x_2, y_4, y_6 \rangle \cap \\ & \langle y_3, y_4, y_6 \rangle \cap \langle x_3, y_5, y_6 \rangle \cap \langle y_4, y_5, y_6 \rangle \,. \end{aligned}$$

One component $\langle z_b \ : \ b \in B \rangle$ for each basis B, where $z_b \in \{x_b, y_b\}$.

For $B = 235$ the component is $\langle y_2, x_3, y_5 \rangle$ because

- 2 is **passive**: I can trade $235 \to 135$ where $1 < 2$
- 3 is **active**: I cannot trade $235 \to 2a5$ where $a < 3$
- 5 is **passive**: I can trade $235 \to 231$ where $1 < 5$

This works in general!



TOPOLOGY

A key ingredient is to understand the initial ideals

$$\mathrm{in}_< I(\widetilde{L}) = \langle x_1y_2y_6,\ x_1y_3y_5y_6,\ x_1y_4y_5y_6,\ x_2y_3y_5,\ x_2y_4y_5,\ x_3y_4\rangle$$

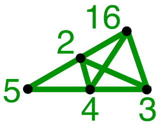
We have

$$\begin{aligned}\mathrm{in}_< I(\widetilde{L}) \quad = \quad & \langle x_1, x_2, x_3\rangle \cap \langle x_1, x_2, y_4\rangle \cap \langle x_1, y_3, y_4\rangle \cap \langle x_1, x_3, y_5\rangle \cap \langle x_1, y_4, y_5\rangle \cap \\ & \langle y_2, y_3, y_4\rangle \cap \langle y_2, x_3, y_5\rangle \cap \langle x_2, x_3, y_6\rangle \cap \langle y_2, y_4, y_5\rangle \cap \langle x_2, y_4, y_6\rangle \cap \\ & \langle y_3, y_4, y_6\rangle \cap \langle x_3, y_5, y_6\rangle \cap \langle y_4, y_5, y_6\rangle .\end{aligned}$$

One component $\langle z_b \ : \ b \in B\rangle$ for each basis B, where $z_b \in \{x_b, y_b\}$.

For $B = 235$ the component is $\langle y_2, x_3, y_5\rangle$ because

- 2 is **passive**: I can trade $235 \to 135$ where $1 < 2$
- 3 is **active**: I cannot trade $235 \to 2a5$ where $a < 3$
- 5 is **passive**: I can trade $235 \to 231$ where $1 < 5$



This works in general!

EXTERNAL ACTIVITY COMPLEX. Equivalently,

Theorem. (A. – Boocher '13)
For any matroid M and any linear order $<$ on the ground set E, there is a simplicial complex $\text{Act}_<(M)$ on $\{x_e, y_e \,:\, e \in E\}$ such that

1. The minimal non-faces are $x_{\min C}\, y_{C - \min C}$ for each circuit C.
2. The facets are the sets $x_{B \cup EP(B)}\, y_{B \cup EA(B)}$ for each basis B.

We call it the **external activity complex** $\text{Act}_<(M)$.

This simplicial complex is **Cohen-Macaulay**. Is it shellable?

Theorem. (A. – Castillo – Samper '14) Yes, it is.
The reduced external activity complex $\overline{Act}_<(M)$ is shellable. It is
- contractible if M contains $U_{3,1}$ as a minor, or
- a sphere if M avoids $U_{3,1}$ as a minor.

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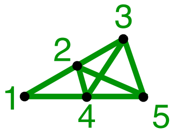
EXTERNAL ACTIVITY COMPLEX. There's many different shellings.

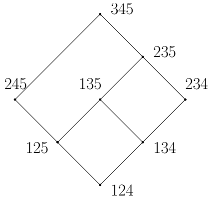
The facets of $\mathrm{Act}_<(M)$ are the sets $x_{B \cup EP(B)} y_{B \cup EA(B)}$ for each basis B.

Las Vergnas defined **external/internal order** $<_{ext/int}$ on the bases:

$A \leq_{ext/int} B$ if and only if $A - IA(A) \cup EA(A) \subseteq B - IA(B) \cup EA(B)$

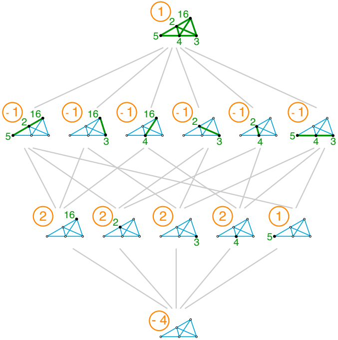
(Motivation: Tutte polynomial, nbc-basis of Orlik-Solomon algebra)





Theorem. (A. – Castillo – Samper '14) Any linear extension of Las Vergnas's order $<_{ext/int}$ gives a shelling of $\overline{Act}_<(M)$.

many thanks



The papers are available at:

http://math.sfsu.edu/federico
http://arxiv.org/