Using CAT(0) cube complexes to move robots efficiently

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Geometric and topological combinatorics:
Modern techniques and methods
MSRI, October 13, 2017

An ongoing research program since 2007 (at MSRI!) with:

- Megan Owen (CUNY), Seth Sullivant (NCSU)
- Rika Yatchak (SFSU → Linz), Tia Baker (SFSU)
- Diego Cifuentes (Los Andes → MIT), Steven Collazos (SFSU → Minnesota)
- Hanner Bastidas (U. Valle), Cesar Ceballos (U. Vienna)
- John Guo (SFSU → UBC) Matt Bland (SFSU) Maxime Pouokam (SFSU → Davis)
- Anastasia Chavez (Berkeley → MSRI → Davis) Arlys Asprilla (ITM)

1. MOTIVATION.

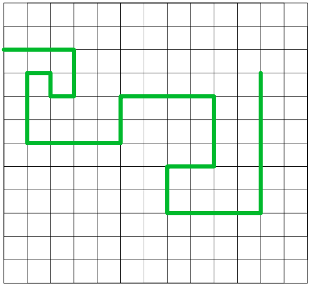
Moving robots.

A robotic snake can move:

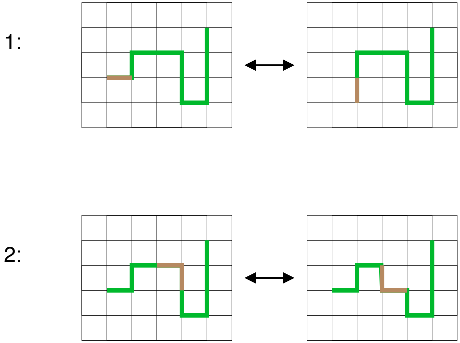
1. the head or tail or 2. a joint

without self-intersecting.

Snake:



Moves:

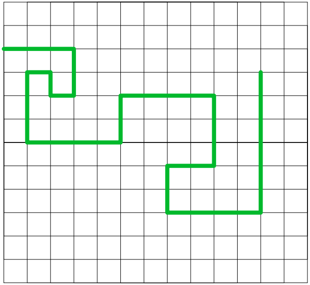


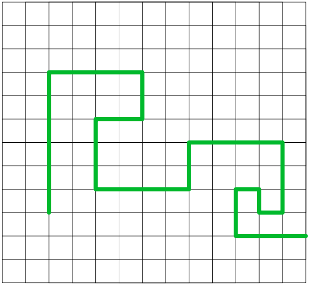
How do we get the robot to navigate this space efficiently?

One motivation: moving robots.

How do can I move this robotic snake (optimally) using these moves from one position to another one?

Position 1 $\rightarrow$ Position 2





Motivation: moving robots.

Well... How do I navigate the world these days?

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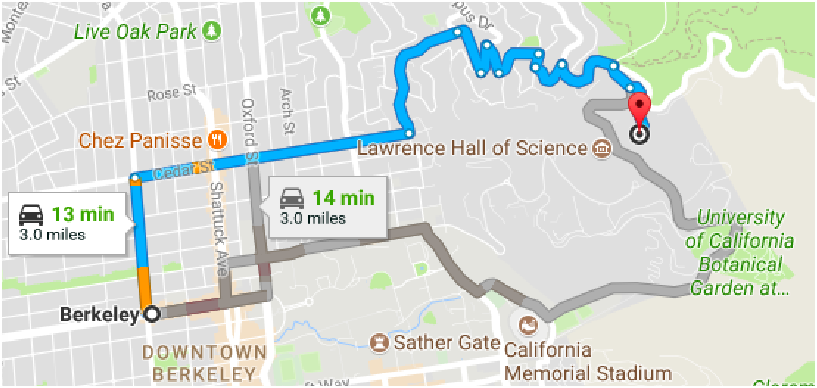
Like this:



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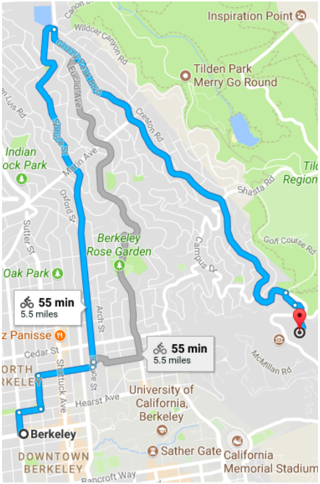
Or like this:
(Q: What does “optimal" mean?)



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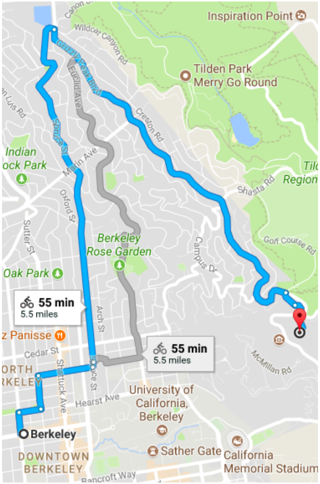
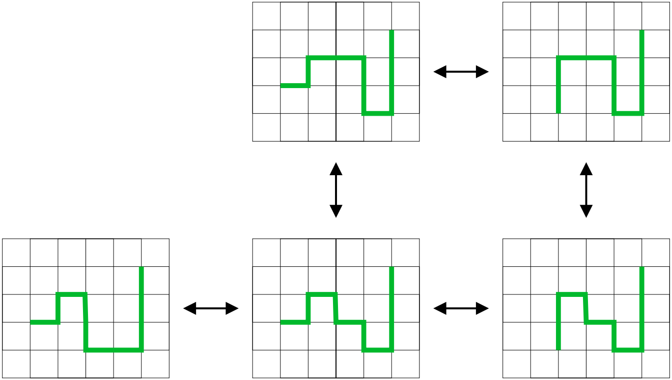


Let’s do the same:
build a map for the robot problem.

One motivation: moving robots.

Let's build a map of all possible positions of the robot.
The moduli space or configuration space.

A small piece: (discrete model)



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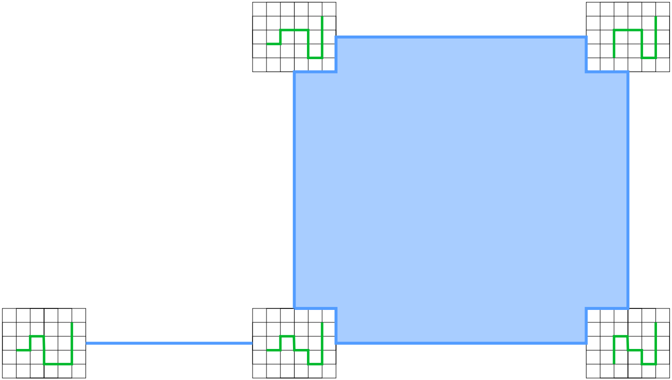
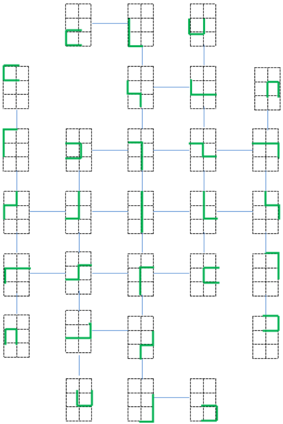
Let's build a map of all possible positions of the robot.

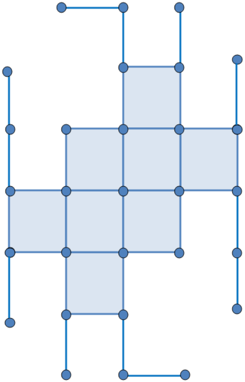
A small piece: (continuous model)



Motivation: moving robots.

Let's build a map of all possible positions. A complete example:





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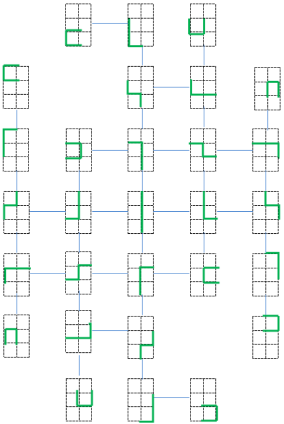
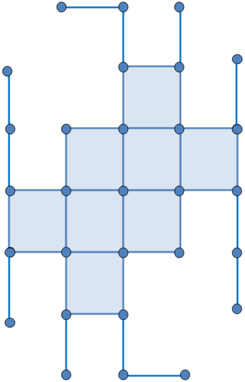
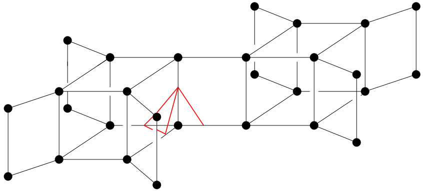


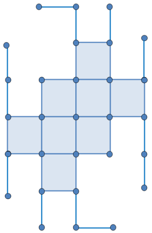


A CAT(0) cube complex!

How can we understand them? Navigate them?

Motivation: moving robots.





How can we understand CAT(0) cube complexes?
How should we navigate them?

Obstacles:

- High dimension.
- Complicated ramification.
- Too many vertices.

This is what we need to overcome.

OK, but before we build a map for the robots...

there are some ethical questions we cannot ignore.

When we were about to submit the paper, this happened:


The Washington Post

The Switch

In an apparent first, Dallas police used a robot to deliver bomb that killed shooting suspect

July 8, 2016

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Very partial thoughts about this:

- Mathematics and science are very powerful tools.
- It is our job to help spread that power equitably.

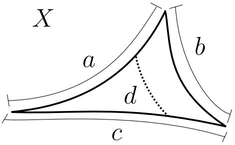
2. PRELIMINARIES. CAT(0) spaces

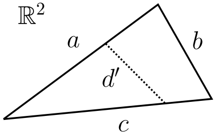
Metric space X is CAT(0) if it has global non-positive curvature. Roughly, it is “saddle shaped".

More precisely triangles in X are “thin". We require:

- There is a unique geodesic path between any two points of X.
- (CAT(0) inequality) Consider any triangle T in X and a *comparison triangle* T' in $\mathbb{R}^2$ of the same sidelengths. Consider any chord (of length d) in T and the corresponding chord (of length d') in T'. Then

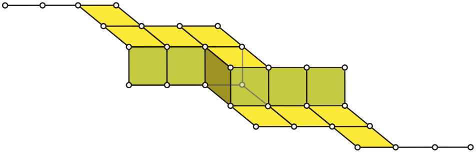
$$d \leq d'.$$





PRELIMINARIES. Cube complexes

A cube complex is a space obtained by gluing cubes (of possibly different dimensions) along their faces.

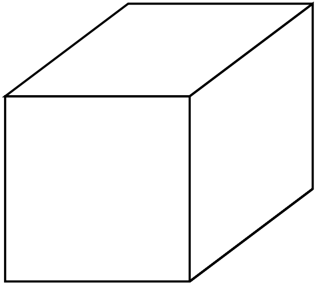


(Like a simplicial complex, but the building blocks are cubes.)

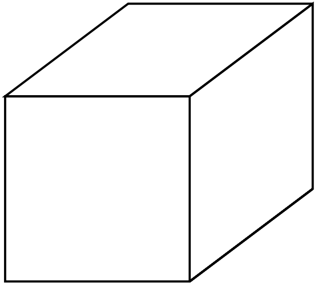
Metric: Euclidean inside each cube.

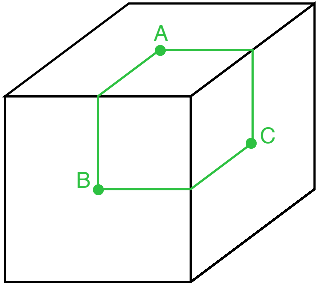
We are interested in cube complexes which are CAT(0).

Example A. The corner of a box. CAT(0)?



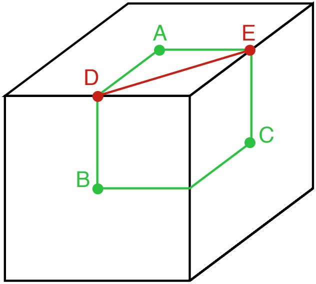
Example A. The corner of a box. CAT(0)?

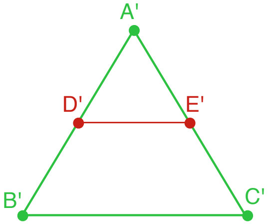




This triangle does not look thin.

Example A. The corner of a box.



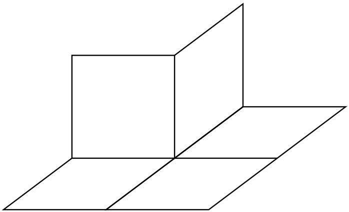


$$|AB| = |BC| = |CA| = 1. \quad \rightarrow \quad |A'B'| = |B'C'| = |C'A'| = 1.$$

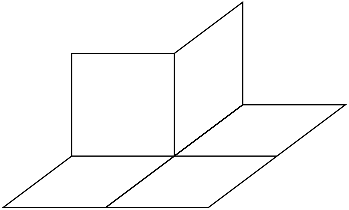
$$|DE| = \frac{\sqrt{2}}{2} \quad > \quad \frac{1}{2} = |D'E'|.$$

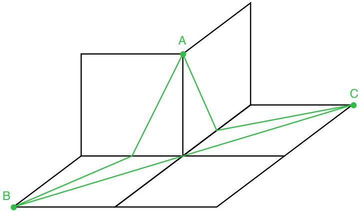
This triangle is not thin. → This space is not CAT(0).

Example B. The corner of a hallway.



Example B. The corner of a hallway.





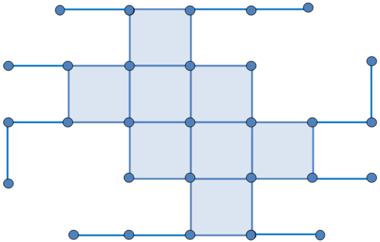
This triangle is thin.
This space IS CAT(0).
(But: I still need to test many triangles.)

This criterion is very impractical!

3. EXAMPLES.

Example 1. Robot motion planning

State complex. vertices = positions. edges = moves.
cubes = “physically independent" moves.



Theorem (Ghrist–Peterson)
This is **often** a CAT(0) cube complex.

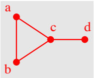
This works **very** generally for many reconfiguration systems, where a discrete system changes according to local moves.

Example 2. Geometric Group Theory. (it started here!)

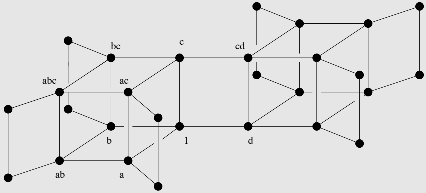
A right-angled Coxeter group is a group of the form

$$W(G) = \langle v \in V \mid v^2 = 1 \text{ for } v \in V, (uv)^2 = 1 \text{ for } uv \in E \rangle$$

Example: $a^2 = b^2 = c^2 = d^2 = 1$
$(ab)^2 = (ac)^2 = (bc)^2 = (cd)^2 = 1$



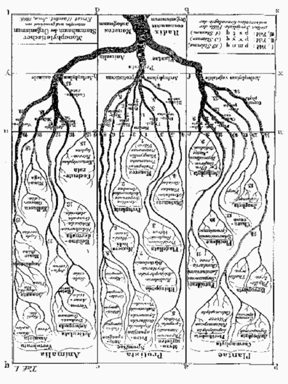
Thm. (Davis) Right-angled Coxeter groups are CAT(0): $W(G)$ acts “very nicely" on a CAT(0) cube complex $X(G)$.



Use the geometry of $X(G)$ to study the group $W(G)$; *e.g.*,
- If a group G is CAT(0), the “word problem" is easy for G.

Example 3. Phylogenetic trees (it started here!)

Goal: Predict the evolutionary tree of
n current-day species/languages/....



Example 3. Phylogenetic trees (it started here!)

Goal: Predict the evolutionary tree of
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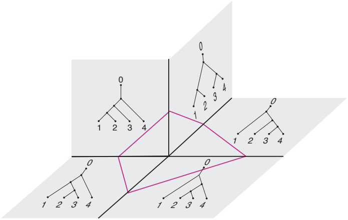
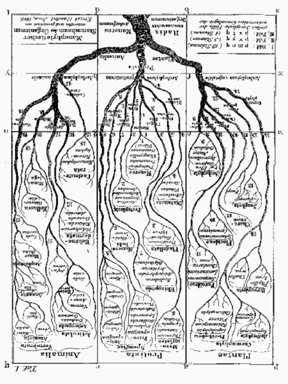
Approach:

- Build a space $\mathbf{T}_n$ of all possible trees.
- Study it, navigate it.

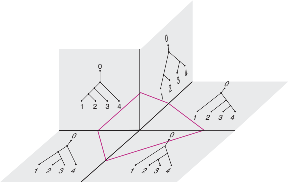


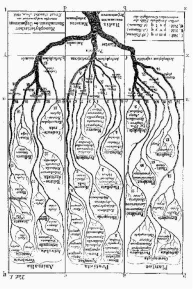


Thm Billera, Holmes, Vogtmann $\mathbf{T}_n$ is a CAT(0) cube complex.	**Cor.** $\mathbf{T}_n$ has unique geodesics. **Cor.** “Average" trees exist.

Example 3. Phylogenetic trees (Billera, Holmes, Vogtmann):

Goal: Predict the evolutionary tree of
n current-day species/languages/....
Idea: Build a space $\mathbf{T}_n$ of all possible trees.





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Related spaces.
algebraic geometry: moduli space $\overline{M_{0,n}}$
topology / geometric group theory: outer space $\mathrm{Out}(F_n)$
tropical geometry: tropical Grassmannian $\mathrm{TropGr}(2, n)$

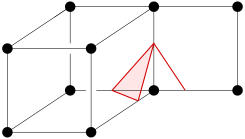
4. CHARACTERIZATIONS.

Which cube complexes are CAT(0)?

In general, CAT(0) is a subtle condition; but for cube complexes:

1. Gromov's characterization.

Theorem. (Gromov, 1987)
A cube complex is CAT(0) if and only if it is simply connected and the link of every vertex is a flag simplicial complex.

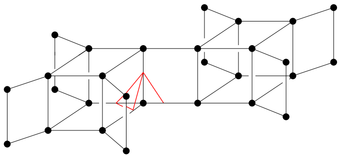


Δ flag: if the 1-skeleton of a simplex T is in Δ, then T is in Δ.

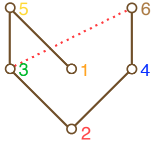
(If a vertex sees the 2-faces of a cube, then the cube is in Δ.)

Characterizations: Which cube complexes are CAT(0)?

2. Our characterization.



Theorem. (A.–Owen–Sullivant 08)
(Pointed) CAT(0) cube complexes are in bijection with posets with inconsistent pairs.



PIP: A poset P and a set of "inconsistent pairs" $\{x, y\}$, with

$$x, y \text{ inconsistent}, \quad y < z \quad \rightarrow \quad x, z \text{ inconsistent.}$$

Theorem. (A.–Owen–Sullivant 08)
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Sketch of proof.

CAT(0) cube complexes “look like" distributive lattices.

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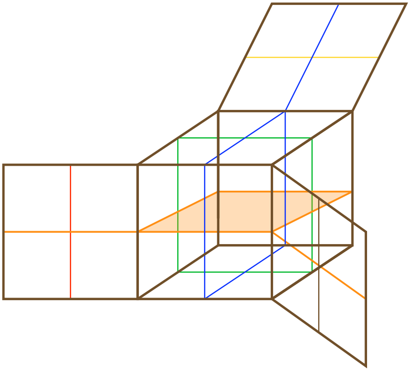
Sketch of proof.

CAT(0) cube complexes “look like" distributive lattices.
So imitate Birkhoff’s bijection: distributive lattices $\leftrightarrow$ posets

“ $\rightarrow$ ”: X has hyperplanes which split cubes in half. (Sageev)



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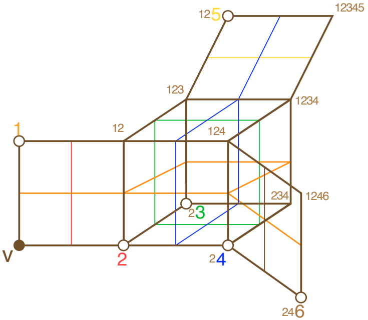
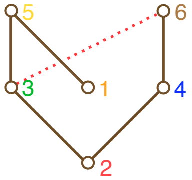
Bijection. " $\rightarrow$ ": Fix a "home" vertex v.





If i, j are hyperplanes, declare:

$i < j$ if one needs to cross i before crossing j
i, j inconsistent if it is impossible to cross them both.

Key Fact: This is enough to recover the cubical complex!

Remark. There are equivalent (and earlier) models:

Computer Science:

WInskel (87): event structure (with binary conflict)

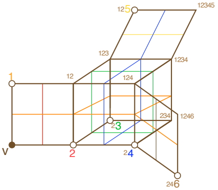
Geometric Group Theory:

Sageev (95) and Roller (98): pocsets

APPLICATION 1: Geometric Group Theory

Embeddability conjecture.

Conjecture. (Niblo, Sageev, Wise) Any d-dimensional interval in a CAT(0) cube complex can be embedded in the cubing $\mathbb{Z}^d$.



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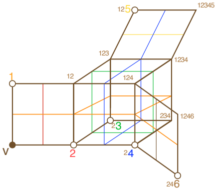

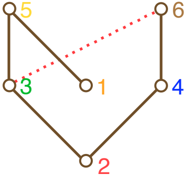

Proof. (AOS 08)

Dilworth already showed (in 1950!) how to embed $J(Q)$ in $\mathbb{Z}^d$:

- Write Q as a union of d disjoint chains. (Example: 246, 35, 1)
- “Straighten" the cube complex along each chain. □

(Proof also by Brodzki, Campbell, Guentner, Niblo, Wright (08).)

APPLICATION 2. Moving CAT(0) robots efficiently.

Two motivations / inspirations:

Geometric Group Theory. (Niblo-Reeves 98)
In a CAT(0) cube complex, the normal cube path finds the shortest cube path between two points.

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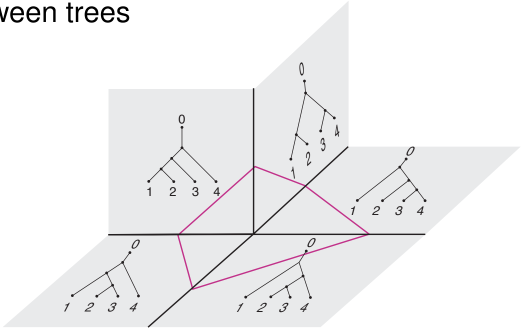
In a CAT(0) cube complex, the normal cube path finds the shortest cube path between two points.

Biostatistics. (Owen-Provan 09) A polynomial-time algorithm to find the geodesic between two trees in the space of trees $\mathbf{T}_n$.

This allows us to

- find distances between trees
- “average" trees.



Moving CAT(0) robots efficiently.

We use the PIP ("remote control") of X to get:

Algorithm. (A.–Owen–Sullivant 12, A.–Baker–Yatchak 14, A.–Bastidas–Ceballos–Guo 16) We can find the geodesic between two points in **any** CAT(0) cube complex X, w.r.t.:

- Time
- Number of moves.
- Number of steps of simultaneous moves.
- Euclidean length

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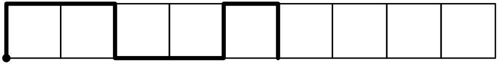
For CAT(0) robots we can find the optimal robotic motion between any two positions.

For non-CAT(0) robots we do not know what to do!
(For example, the robotic snake we started with.)

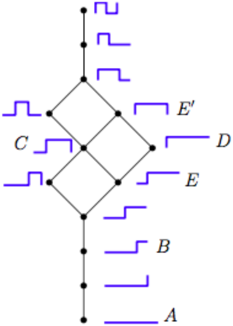
So we should hope our robots are CAT(0)!

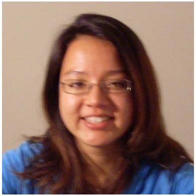
6. MOVING ROBOTS.

Robot 1. A (pinned-down) robotic arm in a tunnel of width 1.



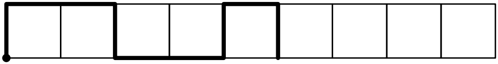
Map for arm of length 5: (A., Tia Baker, Rika Yatchak, 2014)



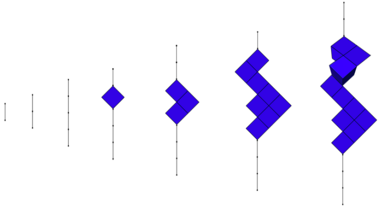


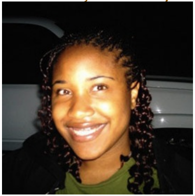
Question. Is it CAT(0)?

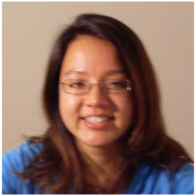
Robot 1. A robotic arm in a tunnel of width 1.



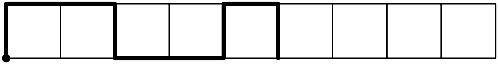
Maps: length 1,2,3,4,5,6,7 (A., Tia Baker, Rika Yatchak, 2014)



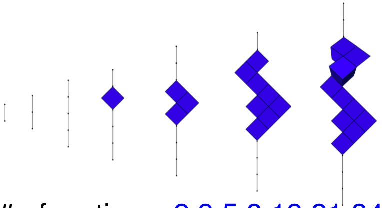


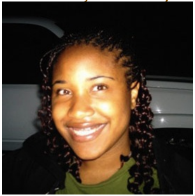


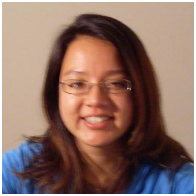
Robot 1. A robotic arm in a tunnel of width 1.



Maps: length 1,2,3,4,5,6,7 (A., Tia Baker, Rika Yatchak, 2014)

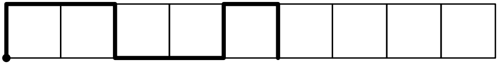




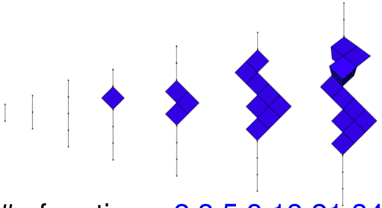


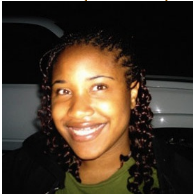
# of vertices: 2,3,5,8,13,21,34,. . .
Fibonacci numbers! Very nice but very large!

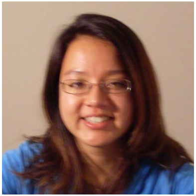
Robot 1. A robotic arm in a tunnel of width 1.



Maps: length 1,2,3,4,5,6,7 (A., Tia Baker, Rika Yatchak, 2014)







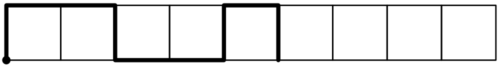
# of vertices: 2,3,5,8,13,21,34,. . .
Fibonacci numbers! Very nice but very large!

For length 100:

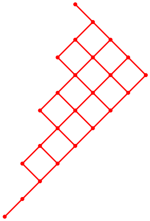
- vertices: 354’ 224,848’ 179,261’ 915,075
- dimension: 34

Without a good idea, navigating these is impossible.

Robot 1. A robotic arm in a tunnel of width 1.



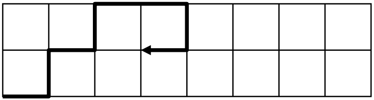
> **Theorem.** (A.-Baker-Yatchak, 2014)
> The state complex is a CAT(0) cubical complex.
> Its PIP (“remote control") is as shown: $\longrightarrow$



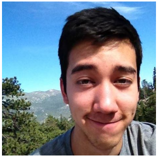
Map: exponential size and linear in dimension.
- 354’ 224,848’ 179,261’ 915,075 vertices, dimension 34

PIP (Remote control): quadratic size and two-dimensional.
- 251,001 vertices, dimension 2

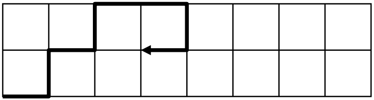
Robot 2. A robotic arm in a tunnel of width 2.



Question. (A.-Bastidas-Ceballos-Guo, 2015)
Is the configuration space a CAT(0) complex?



Robot 2. A robotic arm in a tunnel of width 2.



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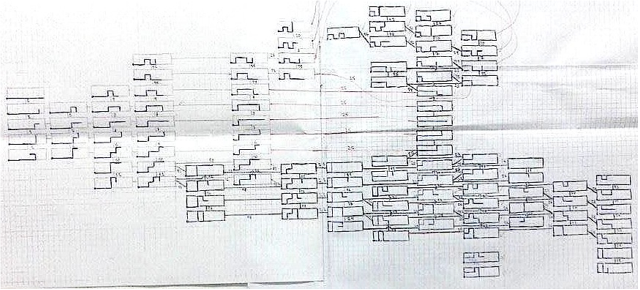
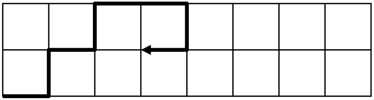
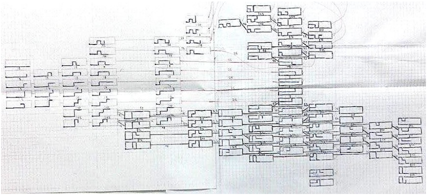


This space is much larger and more complicated.

Robot 2. A robotic arm in a tunnel of width 2.

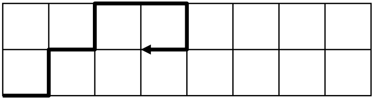


Question. (A.-Bastidas-Ceballos-Guo, 2015)
Is the configuration space a CAT(0) cubical complex?



Preliminary evidence:
Gromov: This space is CAT(0) $\iff$ it is contractible.
Idea: Let's compute the Euler characteristic.

Robot 2.. A robotic arm in a tunnel of width 2.



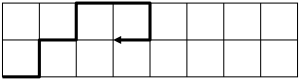
Idea: Let's compute the Euler characteristic.

Preliminary step: the f-vector

Theorem. (A.-Bastidas-Ceballos-Guo, 2015) Let $t_{n,d}$ be the number of d-dimensional cubes in the configuration space for the robotic arm of length n in a tunnel of width 2. Then

$$\sum_{n,d\geq 0} t_{n,d}\, x^n y^d = \frac{1-x+x^2+x^4-x^5+x^2y+x^3y+2x^4y-x^5y+x^4y^2+x^5y^2}{1-2x+x^2-x^3-x^4-2x^4y-2x^5y-x^5y^2-x^6y^2}.$$

Robot 2. A robotic arm in a tunnel of width 2.



Idea: Let's compute the Euler characteristic.

Theorem. (ABCG, 2015) $t_{n,d} = \#$ d-cubes for arm of length n.

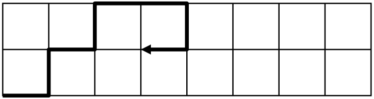
$$\sum_{n,d\geq 0} t_{n,d}\, x^n y^d = \frac{1 - x + x^2 + x^4 - x^5 + x^2y + x^3y + 2x^4y - x^5y + x^4y^2 + x^5y^2}{1 - 2x + x^2 - x^3 - x^4 - 2x^4y - 2x^5y - x^5y^2 - x^6y^2}.$$

Corollary. The configuration space has Euler characteristic 1. (This is the correct Euler characteristic for a CAT(0) space.)

Proof. The Euler characteristic is $t_{n,0} - t_{n,1} + \cdots$ and

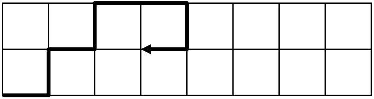
$$\sum_{n,d\geq 0} t_{n,d}\, x^n (-1)^d = \frac{1 - x - x^3 + x^5}{1 - 2x + x^2 - x^3 + x^4 - x^5 - x^6} = \frac{1}{1-x} = 1 + x + x^2 + \ldots.$$

Robot 2. A robotic arm in a tunnel of width 2.



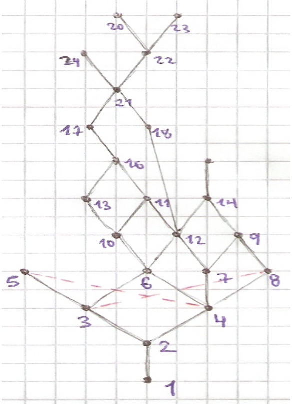
This computation convinced us the space is probably CAT(0).

Robot 2. A robotic arm in a tunnel of width 2.



This computation convinced us the space is probably CAT(0).

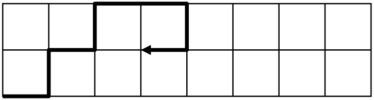
This is the coral PIP for length 6: $\longrightarrow$



How do we describe it in general?

This PIP is much more complicated.

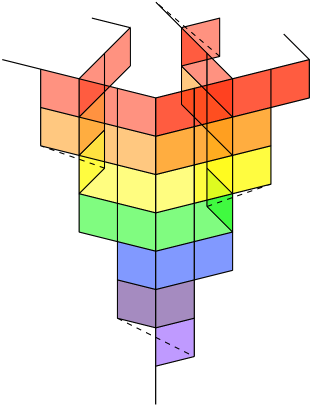
Robot 2. A robotic arm in a tunnel of width 2.



This computation convinced us the space is probably CAT(0).

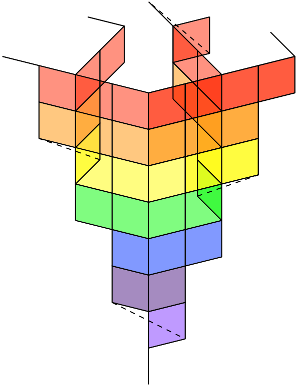
How do we describe the PIP?
A hint came from the Pacific:

Guess. (ABCG, 2015)
The configuration space is CAT(0).
Its PIP is the CORAL PIP $\longrightarrow$



Robot w: A robotic arm in a tunnel of **any width w**.

Theorem. (A. - Bastidas - Ceballos - Guo '16)
For any width, the configuration space of this robot **IS CAT(0)**.
Its PIP is the coral PIP shown. $\longrightarrow$



- Elements of the coral PIP:

Pairs (λ, s) where

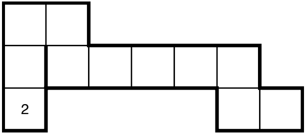
  - λ is a *coral snake* with $h(\lambda) \leq w$
  - $s \in [w(\lambda) - 1, n - l(\lambda)]$

- Order:

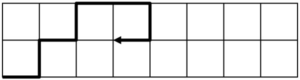
$(\lambda, s) \leq (\mu, t)$ if $\lambda \subseteq \mu$, $s \geq t$.

- Inconsistency:

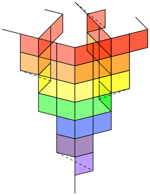
$(\lambda, s) \not\leftrightarrow (\mu, t)$ if $\lambda \not\subset \mu$ and $\lambda \not\supset \mu$



More generally: A robotic arm in a tunnel of **any** width w.



Theorem. (A. - Bastidas - Ceballos - Guo '16)
The configuration space **IS CAT(0)**.
Its PIP is the coral PIP shown: $\longrightarrow$

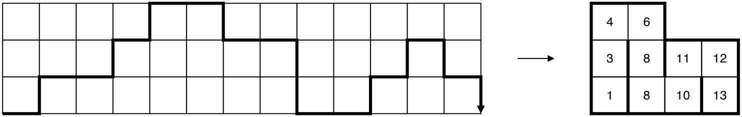


Key Idea: A bijection

states of the arm $\longleftrightarrow$ **coral snake tableau**

A **coral snake tableau** is a filling of λ with integers which are:
- strictly increasing horizontally
- weakly increasing vertically

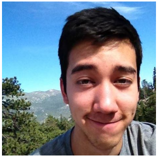
in the direction of the snake.



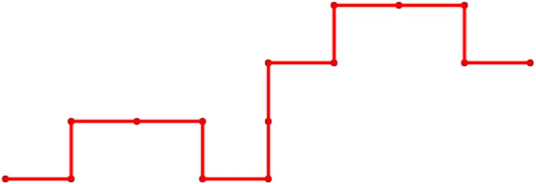
6. SO, HOW DO WE MOVE THE ROBOTS?

These robotic arms are CAT(0); we can move them efficiently!

We have implemented this algorithm in Python:
(FA,Cesar Ceballos,Hanner Bastidas,John Guo,2016)



```
Enter the number of rows in the grid (grid height): 5
Enter a valid state:  ruruurddruururd
Enter a valid state:  rruurdrrrdruruu
The minimum number of steps is 26 .
The minimum number of individual moves is 67.
```



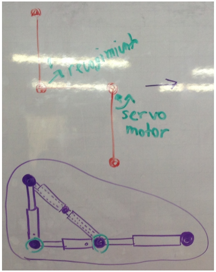
Let's watch a video.

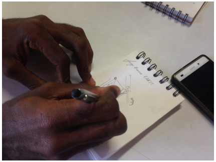
5. SO, HOW DO WE MOVE THE ROBOTS?

Clubes de Ciencia Colombia (July, 2016)

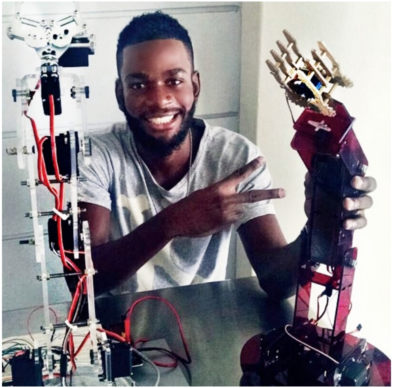
Cesar Ceballos (U. Viena), Olga Salazar (U. Nal. Medellín)

Arlys Asprilla, Cristian Lopez, Daniel Betancur,
Diego Penagos, Dubenis López, Felipe Hoyos,
Juan C. Cuervo, Juan E. Zabala, Juan M. Patiño,
Manuel Ramos, María F. Gualero, Santiago Martínez,
Sebastián Ramírez, Sebastián Sánchez, Wolsey Rubio.



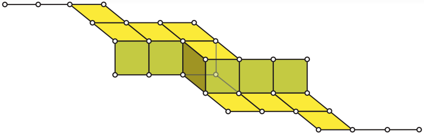


5. SO, HOW DO WE MOVE THE ROBOTS?



Arlys Javier Asprilla
Istmina, Chocó $\longrightarrow$ ITM Medellín, Colombia $\longrightarrow \cdots$

Let's watch another video.



muchas gracias

The articles and slides are at:

Advances in Applied Mathematics **48** (2012) 142-163.
SIAM J. Discrete Math. **28-2** (2014), pp. 986-1007
SIAM J. Discrete Math. (2017) To appear.

http://arxiv.org/

http://math.sfsu.edu/federico