The geometry of geometries: matroid theory, old and new

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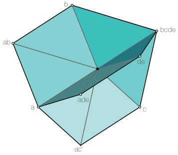

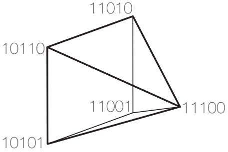

I'll talk about many people's work, including my coauthors, students, and teachers:

Marcelo Aguiar, Carolina Benedetti, Adam Boocher, Federico Castillo, Graham Denham, Jeff Doker, Laura Escobar, Chris Eur, Alex Fink, June Huh, Carly Klivans, Alex Postnikov, Vic Reiner, Felipe Rincón, Gian-Carlo Rota, Mario Sanchez, José Samper, Richard Stanley, Bernd Sturmfels, Mariel Supina, Lauren Williams, and so many others.

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Summary.

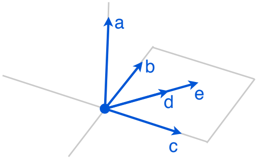
- Matroids are geometric.
- This helps geometry and combinatorics.

Matroids

Goal: Capture the combinatorial essence of independence.

Motivating example:

E= set of vectors in a vector space V
$\mathcal{B}$ = collection of subsets of E which are bases of V



$E = abcde$

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Exchange Property:
If $A, B \in \mathcal{B}$ and $a \in A - B$,
then there exists $b \in B - A$
such that $(A - a) \cup b \in \mathcal{B}$.

Definition. (Nakasawa, Whitney, 1935)
A set E and a collection $\mathcal{B} \neq \emptyset$ of subsets of E form a **matroid** if they satisfy the Exchange Property.

Many matroids in “nature" $\mapsto$ many applications!

Intended examples:

$E = abcde$
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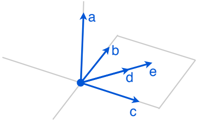
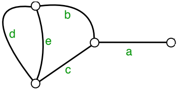
1. Linear algebra
E= set of vectors in V
$\mathcal{B}$ = bases of V in E



2. Graph theory
E= edges of a connected graph G.
$\mathcal{B}$ = spanning trees of G.



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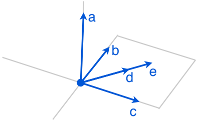

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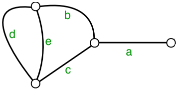

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Unintended examples:

3. Field theory (algebraic independence)
4. Matching theory (e.g. jobs $\leftrightarrow$ people)
5. Routing theory (non-intersecting paths)
...and many, many, many others.

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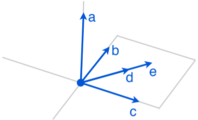

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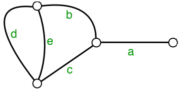

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The unintended examples are equally fundamental to the theory.

Are matroids geometric?

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(linear matroids) vs. (all matroids):

- Some matroids are non-linear. (Whitney, 35)
- 100% of matroids are non-linear. (Nelson, 18)

  (But almost any matroid we think of is linear.)
- Is there a “missing axiom" for linearity? No. (Mayhew et al, 14)

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This is not a flaw!
Non-linear matroids are fundamental to the (geometric) theory.

The geometry of geometries.

My main point.
(All) matroids are natural geometric objects.

> When I wrote my book on 'matroids, I changed the name. I called it "Combinatorial Geometries" – but it didn't take. They said "that's really matroids, isn't it?"

Gian-Carlo Rota, Combinatorial Theory, Fall 1998. (Thanks to John Guidi.)

Today, I will share (a non-exhaustive sample of) four geometric models of matroids. For each one, I will discuss:

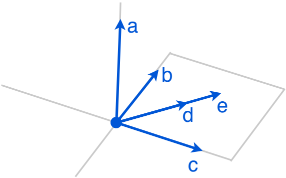
- ◦ Definition
- ◦ Geometric motivation
- ◦ How it helps us understand matroids
- ◦ Applications

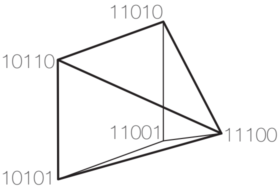
Model 1: Matroid polytopes

Def. (Edmonds 70; Gelfand Goresky MacPherson Serganova 87)
The **matroid polytope** of a matroid M on E is

$$P_M = \operatorname{conv}\{e_B : B \text{ is a basis of } M\} \subset \mathbb{R}^E$$

where e_B is the $0-1$ indicator vector of B.





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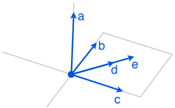
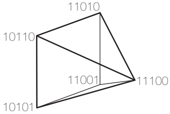
Motivation: Toric Geometry and Optimization.

The matroid polytope of M is $P_M = \mathrm{conv}\{e_B : B \text{ is a basis of } M\}$.





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Matroid polytopes in “nature":

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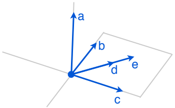

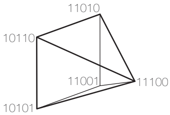

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Matroid polytopes in “nature":

1. Optimization. (Edmonds 70)

For $c : E \to \mathbb{R}$, find the bases $\{b_1, \ldots, b_r\}$ of minimal cost $c(b_1) + \cdots + c(b_r)$.

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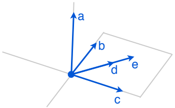
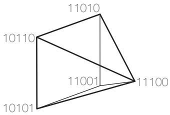
The matroid polytope of M is $P_M = \mathrm{conv}\{e_B : B \text{ is a basis of } M\}$.





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2. Algebraic geometry. (Gelfand Goresky MacPherson Serganova 87)

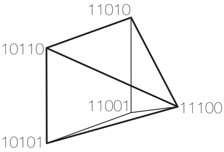
Understand the action of the torus $(\mathbb{C}^*)^n$ on the Grassmannian $\mathrm{Gr}(k, n)$.

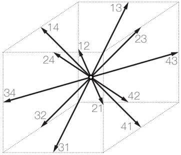
A Lie theoretic characterization of matroids

Theorem. (GGMS 87) A collection $\mathcal{B}$ of r-subsets of $[n]$ is a matroid if and only if every edge of the polytope

$$P_M = \mathrm{conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$$

is a translate of vectors $e_i - e_j$ for some i, j.





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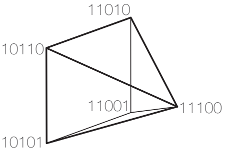

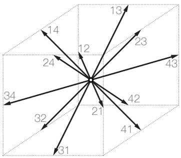

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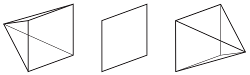
Def. A **matroid** is a 0-1 polytope with edge directions $e_i - e_j$.

From this geometric viewpoint, all matroids are equally natural.
Matroids provide the correct level of generality.

Applications.



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(FA-Benedetti-Doker 10)

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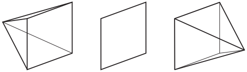

1. Deg(torus orbit in $Gr_{k,n}$) = Vol(matroid polytope).
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2. Matroids form a Hopf algebra via $\oplus$, $/$, $\backslash$. (Joni-Rota 78)
   - antipode(M) = $\pm \sum_{P_N \text{ face of } P_M} (-1)^{\dim(P_N)} N$ (Aguiar-FA 17)

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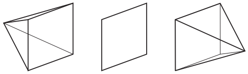

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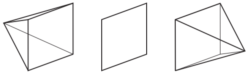

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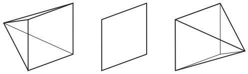

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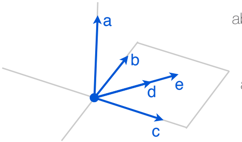
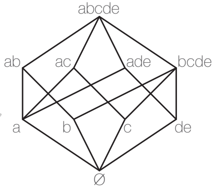
4. Matroid subdivisions: If P_M cannot be subdivided into $P_{M'}$s, then M has finitely many linear representations over any fixed field. (Lafforgue 03)
   - matroid valuations (Speyer 04, FA-Fink-Rincón 07, Derksen-Fink 09)
   - Coxeter matroid valuations (Eur-Sanchez-Supina 19)

Model 2: Matroid fan (Bergman fan)

Flats of a matroid: spanned sets

$\mathcal{F} = \{\emptyset, \quad a, b, c, de, \quad ab, ac, ade, bcde, \quad abcde\}$





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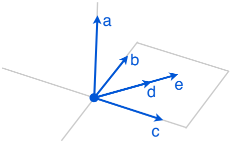
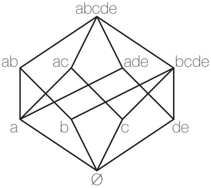
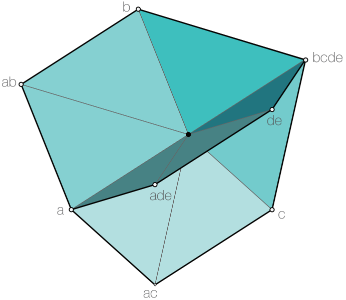






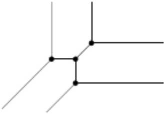
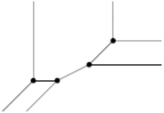
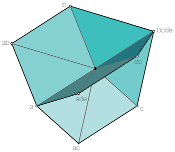
Definition/Theorem. (Sturmfels 02, FA–Klivans 06)
The **matroid fan** Σ_M of a matroid M is the polyhedral fan with

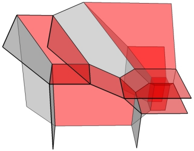
- rays: $e_F := e_{f_1} + \cdots + e_{f_k}$ for each flat $F = \{f_1, \ldots, f_k\}$
- faces: $\text{cone}\{e_F : F \in \mathcal{F}\}$ for each flag $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq E\}$.

Motivation: Tropical Geometry

An algebraic variety V **tropicalizes** to a polyhedral complex Trop(V) that roughly captures its behavior at infinity:

$$\operatorname{Trop} V = \lim_{t\to\infty} \{\log_t(|z_1|,\ldots,|z_n|) : (z_1,\ldots,z_n) \in V\} \quad \text{for} \quad V \subseteq (\mathbb{C}^*)^n.$$



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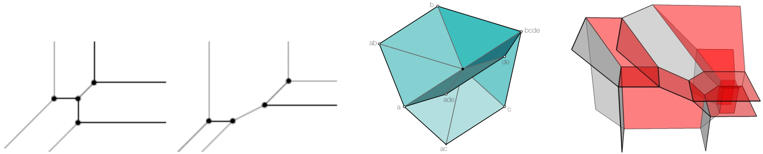

Theorem. (FA–Klivans 06) Let $V \subseteq (\mathbb{C}^*)^n$ be a linear space and $M(V)$ be its matroid. The tropicalization of V is the matroid fan of $M(V)$:

$$\operatorname{Trop}(V) = \Sigma_{M(V)}.$$

It is (the cone over) a wedge of $|\mu(M(V))|$ spheres.

A tropical characterization of matroids

A **tropical fan** is a weighted polyhedral fan with 0-tension**.**

It has a **tropical degree**, and AlgDeg(V) = TropDeg(Trop V)**.**

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A **tropical fan** is a weighted polyhedral fan with 0-tension.

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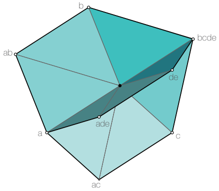
Theorem. (Fink 2013)
A tropical fan has degree 1 if and only if it is a matroid fan.



Definition. A **matroid** is a tropical fan of degree 1.

From this geometric viewpoint, all matroids are equally natural.
Matroids provide the correct level of generality!

Applications.

1. The Chow ring of the Bergman fan Σ_M satisfies the **Kähler package**, as if it were the cohomology ring of a smooth projective variety. (!!!) This implies:
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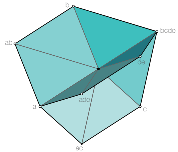

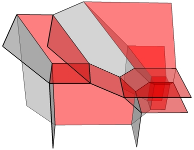

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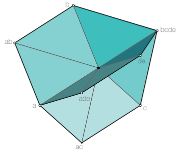

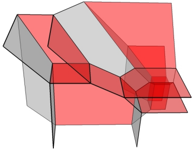

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3. **Chern–Schwartz–MacPherson** classes of matroids and tropical manifolds.
(López de Medrano–Rincón–Shaw, 2020)

Model 3: The conormal fan

Theorem. If $\mathcal{B}$ is a matroid on E, then $\mathcal{B}^\perp = \{E - B : B \in \mathcal{B}\}$ is also a matroid on E, the **orthogonal** or **dual** matroid $M^\perp$.

This generalizes:

- Dual graphs:

  abe spanning tree of G

  $\updownarrow$

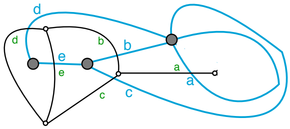
  cd spanning tree of G^*

- Orthogonal complements:

  abe basis of W

  $\updownarrow$

  cd basis of $W^\perp$



$$W = \text{rowspace}\begin{bmatrix} 0 & 1 & 0 & .5 & 1 \\ 0 & 0 & 1 & .5 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W^\perp = \text{rowspace}\begin{bmatrix} 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix}$$

Definition. (FA-Denham-Huh, 16)
The **conormal fan** $\Sigma_{M,M^\perp}$ of M is (a certain rather subtle subdivision of) $\Sigma_M \times \Sigma_{M^\perp}$.

Motivation: Lagrangian geometry

Varchenko's **variety of critical points** of V is

$$\begin{array}{rcl} \mathfrak{X}(V) = \text{closure of } \mathbb{P}V \times \mathbb{P}V^{\perp} & \subseteq & \mathbb{CP}^{n-1} \times \mathbb{CP}^{n-1} \\ (x, y) & \longmapsto & (x, xy) \end{array}$$

Closely related:
- *incidence varieties* in projective geometry,
- *conormal varieties* in Lagrangian geometry,
- *maximum likelihood degrees* in algebraic statistics.

Theorem. (FA–Denham–Huh 17)
Let $\mathbb{P}V \subseteq \mathbb{CP}^{n-1}$ be a linear space and $M(V)$ its matroid. The variety of critical points of V tropicalizes to the conormal fan of $M(V)$:

$$\operatorname{Trop}(\mathfrak{X}(V)) = \Sigma_{M(V),M(V)^{\perp}}.$$

A Lagrangian characterization of matroids?

We don't know one yet.

Applications.

1. The Chow ring of the conormal fan $\Sigma_{M,M^\perp}$ satisfies the **Kähler package** as well. (!!!) Proving it requires significant extra work. It implies:
$\longmapsto$ ***h*-vector of $BC(M)$ is log-concave** (Brylawski 82 $\mapsto$ FA-Denham-Huh 22)

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4. Two new polytopes with an elegant combinatorial structure.
   - **Harmonic polytope** (FA-Escobar 21)
     $3^n - 3$ facets, $(2n)!/2^n$ vertices
   - **Bipermutahedron** (FA 22)
     $3^n - 3$ facets, $n!(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$ vertices

Model 4: The augmented matroid / Bergman fan

Definition. (FA-Denham-Huh, 2016)
The **augmented matroid fan** is the fan in $\mathbb{R}^{E \sqcup E}$ with

- rays: d_i for each $i \in E$, e_F for each flat F
- $\text{cone}(I, \mathcal{F}) := \text{cone}(\{d_i\}_{i \in I} \cup \{e_F\}_{F \in \mathcal{F}})$ for I independent, $\mathcal{F}$ flag such that $I \subset F$ for all $F \in \mathcal{F}$.

Analogously to the matroid fan,

Theorem (Bullock-Kelley-Reiner-Ren-Shemy-Shen-Sun-Tao-Zhang 21)
The augmented Bergman fan of M is (the cone over) a wedge of $|\mathcal{B}(M))|$ spheres.

Motivation: Schubert varieties of matroids

The **Schubert variety of a linear space** $V \subset \mathbb{C}^n$ is

$$Y(V) = \text{closure of } V \text{ in embedding } \mathbb{C}^n \hookrightarrow (\mathbb{P}^1)^n.$$

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The augmented Bergman fan is a tropical analog of $Y(V)$.

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Intersection cohomology $IH(M)$ **of** M:
- M linear: study $Y(V)$ and its resolution of singularities.

(Huh-Wang 17)
- M general: extend the definition above combinatorially.

(Braden-Huh-Matherne-Proudfoot-Wang 22)

An augmented characterization of matroids?

We don't know one yet.

Applications.

1. The **intersection cohomology of** M satisfies the **Kähler package** as well.
(!!!) Proving it requires significant extra work. It implies:
$\longmapsto$ **lattice of flats of** M **is top heavy**

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3. Gröbner theory leads to rich **topological combinatorics of matroids**.
(FA-Castillo-Samper 16)

Geometry and Combinatorics. Two visionary remarks.

example is so beautiful that we decided to publish it independently of the applications. We believe that combinatorial methods will play an increasing role in the future of geometry and topology.

Gelfand–Goresky–MacPherson–Serganova, 1987

of dedication and lasting achievements, we were struck by one remark, which to our minds was later to prove prophetic: "We combinatorialists have much to gain from the study of algebraic geometry, if not by its direct applications to our field, at least by the analogies between the two subjects."

R. C. Bose (quoted by Kelly–Rota, 1973)

Today these remarks ring true more than ever.

many thanks

muchas gracias

- More details and references are in my Proceedings of ICM contribution.
- These slides are on my website.