Lecture Lecture 1 *43 min.* This is the first lecture of the Spring 2008 course on Coxeter groups, taught jointly at San Francisco State University and the Universidad de Los Andes in Bogota, Colombia. We discuss three ways of thinking about the symmetric group S_3: combinatorially in terms of permutations or wiring diagrams, algebraically in terms of generators and relations, and geometrically in terms of reflections.

Lecture Lecture 2 *51 min.* We define Coxeter systems and Coxeter groups, and discuss several interesting examples

Lecture Lecture 3 *47 min.* We introduce a concrete combinatorial realization of an arbitrary Coxeter group (W,S) - its "signed permutation representation".

Lecture Lecture 4 *53 min.* We prove the signed permutation presentation of a Coxeter group. We pay special attention to the symmetric group, where wiring diagrams make this construction very explicit.

Lecture Lecture 5 *49 min.* We proved that the length of w in (W, S) can be read off from the signed permutation presentation of $\pi(W)$. It equals the number of reflections that $\pi_{w^{-1}}$ makes negative. These are the reflections t that shorten w : $l(tw) < l(w)$. For the symmetric group, these are the inversions of the permutation w.

Lecture Lecture 6 *50 min.* We prove the exchange and deletion properties for Coxeter groups, and several useful consequences of them.

Lecture Lecture 7 *55 min.* We prove that for a group W and a set S of generators of order 2, the following three conditions are equivalent: (W,S) is a Coxeter system, it satisfies the exchange property, and it satisfies the deletion property. We use this to prove that the symmetric group is a Coxeter group.

Lecture Lecture 8 *49 min.* We discuss the geometric representation of a Coxeter group in terms of reflections, and use it to prove two important facts about a Coxeter system (W,S) with matrix m: the order of ss' is m(s,s'), and S is a minimal set of generators for W.

Lecture Lecture 9 *49 min.* We introduce the Bruhat order of a Coxeter group. For the symmetric group, this order describes "how special" the relative positions of two flags is.

Lecture Lecture 10 *54 min.* We prove that the Bruhat order is graded, and that $u < v$ if and only if a reduced word for v contains a reduced word for u as a subword.

Lecture Lecture 11 *43 min.* We prove the "lifting property" of the Bruhat order. We use it to conclude that any finite Coxeter group has a unique maximal element, whose properties we study.

Lecture Lecture 12 *50 min.* We prove that multiplication by the longest element of a finite Coxeter group reverses the Bruhat order. Then we define parabolic subgroups and show they are Coxeter groups. (Note. The microphone is off for the first 12 minutes.)

Lecture Lecture 13 *53 min.* We introduce and describe the parabolic quotient W^J, which consists of the unique minimum elements in each of the left cosets of the parabolic subgroup W_J.

Lecture Lecture 14 *46 min.* We compute the Poincare series of the symmetric group, and present a recursive technique for computing the Poincare series of an arbitrary Coxeter group.

Lecture Lecture 15 *54 min.* We introduce the Mobius function of a poset. We define Eulerian posets; two important examples are the face poset of a polytope and any interval in a Bruhat order.

Lecture Lecture 16 *47 min.* We introduce the weak order of a Coxeter group and some of its properties.

Lecture Lecture 17 *50 min.* We prove that the weak order is a meet-semilattice.

Lecture Lecture 18 *52 min.* We discuss various properties of the weak order of a Coxeter group. We illustrate how the weak order has a very nice and useful geometric interpretation.

Lecture Lecture 19 *52 min.* We prove the geometric representation of a Coxeter group. (Note. The proof of Lecture 8 had some serious mistakes - my apologies.)

Lecture Lecture 20 *50 min.* We define root systems and explain how W acts on them. If α is the simple root corresponding to the generator s, we show that $w\alpha > 0$ if $l(ws) > l(w)$ and $w\alpha < 0$ if $l(ws) < l(w)$.

Lecture Lecture 21 *47 min.* We prove that w send exactly l(w) positive roots to negative roots in the geometric representation. Then we show the bijective correspondence between positive roots, algebraic reflections in W, and geometric reflections in the geometric representation of W.

Lecture Lecture 22 *46 min.* If α is the positive root corresponding to the reflection t, we show that $w\alpha > 0$ if and only if $\ell(wt) > \ell(w)$. We then discuss the reflection representation of a finite Coxeter group, and the fundamental domain.

Lecture Lecture 23 *52 min.* We describe the subgroup of W fixing a point or a subset of V in the reflection representation. For W finite, the hyperplanes orthogonal to the roots divide V into chambers which correspond to the elements of W, and the faces of the arrangement are labeled as conjugates of the faces of the fundamental chamber.

Lecture Lecture 24 *53 min.* We define the Coxeter complex of W and show how it can be labelled by the left cosets of the parabolic subgroups of W. Then we talk about the complications that arise in the reflection representation of an infinite W, and how to solve them using the Tits cone.

Lecture Lecture 25 *50 min.* We define the depth of a root, and the root poset. We briefly discuss the W-Catalan numbers.

Lecture Lecture 26 *51 min.* We define small roots and characterize them as those which do not dominate any other roots.

Lecture Lecture 27 *41 min.* We prove that small roots form an order ideal in the root poset. We show that in finite Coxeter groups every root is small. We state the theorem that any finitely generated Coxeter group has finitely many small roots.

Lecture Lecture 28 *50 min.* We construct a finite automaton which recognizes the language of reduced words of any finitely generated Coxeter group.

Lecture Lecture 29 *50 min.* We finish the proof of automaticity of Coxeter groups. We discuss the transfer matrix method for counting paths in graphs. We conclude that the generating function for reduced words in a Coxeter group is rational.

Lecture Lecture 30 *53 min.* We define root systems and give various examples.

Lecture Lecture 31 *48 min.* We prove that a root system gives rise to a Coxeter system.

Lecture Lecture 32 *53 min.* We introduce crystallographic root systems, and show that they correspond to integer Cartan matrices.

Lecture Lecture 33 *52 min.* We describe the Coxeter group corresponding to a crystallographic root system.

Lecture Lecture 34 *52 min.* We characterize integer Cartan matrices, or equivalently crystallographic root systems.

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Lecture Lecture 35 *51 min*. We characterize the Coxeter groups that correspond to crystallographic root systems.
Lecture Lecture 36 *51 min*. We review some facts about bilinear forms. We begin to prove that a Coxeter group is finite if and only if its associated bilinear form is positive definite.
Lecture Lecture 37 | Notes *52 min*. We prove some basic facts about the representation theory of finite groups. (The camera motor is off - please see the lecture notes.)
Lecture Lecture 38 *50 min*. We complete the proof of the theorem that W is finite if and only if its associated bilinear form is positive definite.
Lecture Lecture 39 *50 min*. We prove two characterizations of positive definite symmetric matrices.
Lecture Lecture 40 *50 min*. We begin the classification of finite Coxeter groups.
Lecture Lecture 41 *45 min*. We complete the classification of finite Coxeter groups.
Lecture Lecture 42 *42 min*. We classify the regular polytopes.