Pf (a) $L(P)$ has a meet: $F \wedge G = F \cap G$.

Any poset with a meet and a $\hat{1}$ has a join:

$$F \vee G = \bigwedge(\text{upper bounds of } F \text{ and } G)$$

(non-empty)

Graded: soon

(b)

$[\hat{0}, G]$ is $L(G)$

$[v_1, G]$ is $L(G/v_1)$

$[v_2, G]$ is $L((G/v_1)/v_2)$

$\vdots$

$[F, G]$ is $L((G/v_1)/v_2) \ldots /F)$ → call this polytope G/F

Note: $\dim G/F = \dim G - \dim F - 1$

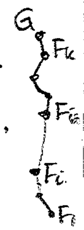
(a) Graded:

Consider a max'l chain $F < F_1 < \ldots < F_k < G$.

Sup I skipped a dimension between F_i, F_{i+1}.

Then $[F_i, F_{i+1}]$ is $L(F_{i+1}/F_i)$ of $\dim \geq 1$

So F_{i+1}/F_i has a vertex. Add it to chain!



(c) Follows from (b) since

$$[F, G] = L(\text{1-D polytope}) = L(\longleftrightarrow) = \Diamond$$

(d) Follows from "polarity", our next goal.

Remark

P and Q are combinatorially isomorphic ($P \simeq Q$) if their face lattices are isomorphic.

Note: This theorem puts very rigid requirements on $L(P)$. These posets have a lot of structure! (More on hw)

Polar Polytopes (also called "dual polytopes")

$P \subset \mathbb{R}^d$ polytope, or any set

The polar of P is

(really the dual space $(\mathbb{R}^d)^*$)

$$P^\Delta = \{c \in (\mathbb{R}^d)^* \mid c \cdot x \leq 1 \text{ for all } x \in P\} \subseteq (\mathbb{R}^d)^*$$

Theorem P, Q polytopes,

(a) $P \subseteq Q \Rightarrow P^\Delta \supseteq Q^\Delta$

(b) $P \subseteq P^{\Delta\Delta}$

(c) If $0 \in P$ then $P = P^{\Delta\Delta}$

(d) If $P = \operatorname{conv} V$ then

$P^\Delta = \{a : a \cdot v \leq 1 \ \forall v \in V\}$

(e) If $P = P(A, 1)$

$P^\Delta = \{cA : c \geq 0, c \cdot 1 = 1\} = \operatorname{conv}(\text{rows } A)$

$P =$ 

$P^\Delta =$ 

$P^{\Delta\Delta} =$ 

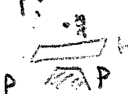
Sketch. (a), (b) are easy.

(c) Sup $q \in P^{\Delta\Delta}$ but $q \notin P$

$c \in P^\Delta \Rightarrow c \cdot q \leq 1$

↳ Let $c \cdot x = c_0$ be a hyp. separating q from P:

$$\begin{cases} c \cdot q > c_0 \\ c \cdot p < c_0 \text{ for } p \in P \end{cases}$$



Since $0 \in P$, $c_0 > 0$. So

$$\frac{c}{c_0} \in P^\Delta \leadsto \begin{cases} \frac{c}{c_0} \cdot q > 1 \\ \frac{c}{c_0} \cdot p < 1 \text{ for } p \in P. \end{cases} \quad \blacksquare$$

(d) $P^\Delta = \{a : a \cdot p \leq 1 \ \forall p \in P\} = \{a : a \cdot v \leq 1 \ \forall v \in V\}$

$\subseteq$: trivial

$\supseteq$: $a \cdot x$ achieves its max at a vertex,