## Other "obvious" facts. (Proofs in the book)

**Prop** $P$ polytope, $V = \mathrm{vert}(P)$

(i) Any face $F$ is a polytope and $\mathrm{vert}(F) = V \cap F$

(ii) $F, G$ faces $\Rightarrow F \cap G$ face

(iii) $F$ face $\Rightarrow$ (faces of $F$) = (faces of $P$ contained in $F$)

(iv) $F$ face $\Rightarrow F = P \cap \mathrm{aff}(F)$

Sketches: (interesting parts).

(ii) $F = P_b,\ G = P_c \Rightarrow F \cap G = P_{b+c}$

(iii) $\supseteq$: clear

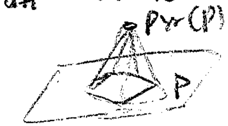
$\subseteq$: $F = P_b,\ G = F_c \Rightarrow G = P_{b+\varepsilon c}$ for $\varepsilon$ small enough.

Lecture 8
Sep 13, 10

## Two more constructions

Pyramids: $P \subset \mathbb{R}^d \longrightarrow$ embed as hyp. $x_{d+1} = 0$ in $\mathbb{R}^{d+1}$

$\mathrm{pyr}(P) = \mathrm{conv}\left\{ \binom{p}{0} : p \in P \right\} \cup e_{d+1}$

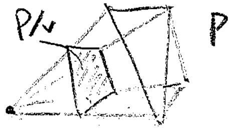



Combin. type of $P$ dets. that of $\mathrm{pyr}(P)$ (HW)

Vertex Figures $P \subset \mathbb{R}^d$, $v$ vertex, say $v = P_c$ $c \cdot v = c_0$

Choose $c_1 < c_0$ (but very close to $c_0$), let

$P/v = P \cap \{x : c \cdot v = c_1\}$




Combin type of $P, v$ dets that of $P/v$.

In fact:

k-faces of $P/v$ $\longleftrightarrow$ $(k+1)$-faces of $P$ containing $v$

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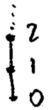
## The Face lattice

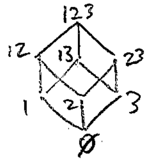
Recall: A poset $(P, \leq)$ is a set $P$ with a partial order such that

- $x \leq x$
- $x \leq y,\ y \leq z \Rightarrow x \leq z$
- $x \leq y,\ y \leq x \Rightarrow x = y$

Ex: $(\mathbb{N}, \leq)$




"Boolean" poset $B_n = (2^{[n]}, \leq)$




Face poset $L(P) = (\text{faces of } P, \subseteq)$

- $L(\square) =$ 
- $L(\Delta_{d-1}) = B_d$ (Exercise)

Chain: $p_1 < p_2 < \cdots < p_k$ in $P$ (length $= k-1$)

$P$ is graded if all maxl chains from $a$ to $b$ have the same length

(graded: $B_n$ not graded: )

If graded, it has "levels" or "ranks"

$P$ is a lattice if any $x, y \in P$ have a least upper bound ("meet" $x \vee y$) and a greatest lower bound ("join" $x \wedge y$).

(lattice: $B_n$ not lattice: )

**Theorem** $P$ polytope

(a) $L(P)$ is a lattice graded by $\mathrm{rk}(F) = \dim F + 1$

(b) Every interval $[F, G]$ is also a face lattice.

(c) Every interval of length 2 is a diamond: 

(d) The opposite poset $L(P)^{op}$ is also a face lattice.

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