FACES OF POLYTOPES (Ziegler, Ch. 2)

Lecture 7
Sep 10, 10

$P \subset \mathbb{R}^d$ polytope

$c \in \mathbb{R}^d$ vector

The face of P in direction c is

$$P_c = \{x \in P : c\cdot x \text{ is maximal}\}$$

Note: P bounded $\to$ $c\cdot x$ bounded, say $c\cdot x \le c_0$ ← least upper bound

$$P_c = \{x \in P : c\cdot x = c_0\}$$

Remark: Affine spaces

- An affine subspace of $\mathbb{R}^d$ is given by:
  - $\{x \in \mathbb{R}^d : Ax = b\}$
  - translate of a linear subspace
  - affine span of a set V

$$\mathrm{aff}(V) = \{x : x = \underbrace{\lambda_1 v_1 + \dots + \lambda_k v_k}_{\text{"affine"}} \text{ for some } v_1, \dots, v_k \in V,\ \lambda_1 + \dots + \lambda_k = 1\}$$

- $V = \{v_1, \dots, v_k\}$ affinely indep. if no v_i is an aff. comb. of $V - v_i$
- dim(affine space) = (size of largest aff indep set)

The dim of a face F is $\dim(\mathrm{aff}(F))$

dim 0: vertex — codim 1: facet

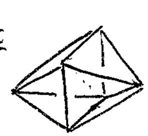
dim 1: edge — codim 2: ridge

The f-vector of P is $(f_{-1}, f_0, \dots, f_d) = f(P)$ where

f_i = # of i-faces of P

The f-poly is $f_P(x) = \sum_0^d f_i x^i$

Ex



$f(P) = (1, 6, 12, 8, 1)$

$f_P(x) = 6 + 12x + 8x^2 + x^3$

⚠ Others use slightly different definitions

Ex. d-cube $C_d = \mathrm{conv}(\{-1,1\}^d) = \{x : -1 \le x_i \le 1 \text{ all } i\}$

$v \in \mathbb{R}^d \to (C_d)_v : \{x \in [-1,1]^d : v\cdot x = v_1x_1 + \dots + v_dx_d \text{ max}\}$

$v_i > 0 \to$ need $x_i = 1$ to max $v_i x_i$

$v_i < 0 \to$ need $x_i = -1$

$v_i = 0 \to$ any x_i

So if $v = (>0, <0, 0, 0, <0, >0, 0, >0, 0)$

$$(C_d)_v = \{(1, -1, *, *, -1, 1, *, 1, *)\} \cong C_4$$

Therefore

$$(\text{faces of } C_d) \longleftrightarrow d\text{-tuples in } \{-1, 1, *\}^d$$

$$\dim = \text{number of } *$$

So $f_k = \binom{d}{k} 2^{d-k}$ $k \ge 0$

$$f_{C_d}(x) = \sum_0^d \binom{d}{k} 2^{d-k} x^k = (2+x)^d$$

There are many "obvious" things to prove:

$\boxed{\textbf{Prop } P \text{ polytope} \Rightarrow P = \mathrm{conv}(\mathrm{vertices}(P))}$

Pf Write $P = \mathrm{conv}(V)$, V finite.

If any $v \in V$ is a conv. combin. of $V - v$, delete it.

Repeat until no longer possible, get $P = \mathrm{conv}(W)$.

Claim: $W = \mathrm{vert}(V)$.

$\supseteq$: Sup vertex $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ $v_1 \dots v_n \in P$, $\lambda_i \ge 0$, $\lambda_1 + \dots + \lambda_n = 1$

This contradicts $c\cdot v = c_0$, $c\cdot v_i < c_0$ (all i)

$\subseteq$: Let $w \in W$, $W' = W - w$. Then

$w \notin \mathrm{conv}\, W' \Rightarrow \nexists\, t \ge 0 : w = W't,\ \mathbb{1}t = 1$

$\Rightarrow \nexists\, t \ge 0 : \binom{\mathbb{1}}{W'} t = \binom{1}{w}$

Farkas Lemma 2 $\Rightarrow \exists\, a : a\binom{\mathbb{1}}{W'} \ge 0 \quad a\binom{1}{w} < 0$

$\Rightarrow \exists\, (\beta, -b) : \beta\mathbb{1} - bW' \ge 0,\ \beta - bw < 0$

$\Rightarrow \exists\, b, \beta : bW' \le (\beta, \dots, \beta),\ bw > \beta$

$\Rightarrow w = \text{vertex } P_b$ ∎