Now we let the hard work of the Main Theorem pay off. Many useful consequences!

Farkas Lemma IV

Consider $\underset{n}{d\,\boxed{V}}$, $\underset{n'}{d\,\boxed{Y}}$, $\underset{1}{d\,\boxed{x}}$

Either $\exists$ $\underset{n}{\boxed{t}}$, $\underset{n'}{\boxed{u}}$ s.t. $\mathbb{1}\cdot t = 1$, $t \geq 0$, $u \geq 0$ and $x = Vt + Yu$

or $\exists$ $\underset{d}{\boxed{a}}$, α s.t. $a\cdot v_i \leq \alpha$ (all i) $\quad a\cdot y_j \leq 0$ (all j) but $ax > \alpha$

but not both

Translation: Let $P = \operatorname{conv}(V) + \operatorname{cone}(Y) = \{x : \exists\, t, u, \dots\ \}$
$= P(A, z) = \{x : Ax \leq z\}$

Either $x \in P$
or some linear inequality certifies that $x \notin P$

Pf. If $x \in P$:

Consider any a, α s.t. $a\cdot v_i \leq \alpha \quad a\cdot y_j \leq 0$

Since $x = t_1 v_1 + \cdots + t_n v_n + u_1 y_1 + \cdots + u_n y_n$,

$$a\cdot x = a\cdot(t_1 v_1) + \cdots + a\cdot(t_n v_n) + a\cdot(u_1 y_1) + \cdots + a\cdot(u_n y_n)$$
$$\leq t_1\alpha + \cdots + t_n\alpha + 0 + \cdots + 0 = \alpha \quad ∎$$

If $x \notin P$

Then x must not satisfy a defining ineq $a\cdot\_ \leq \alpha$

So $a\cdot x > \alpha$

But $v_i \in P \Rightarrow a\cdot v_i \leq \alpha$

$v_i + \lambda u_j \in P$ (any $\lambda > 0$) $\Rightarrow (a\cdot v_i) + \lambda(a\cdot u_j) \leq \alpha$ (any $\lambda > 0$)

$\Rightarrow a\cdot u_j \leq 0.$ ∎

Farkas Lemma II

Consider $\underset{d}{m\,\boxed{A}}$, $\underset{1}{m\,\boxed{z}}$.

Either $\exists$ $\underset{1}{d\,\boxed{x}}$ with $Ax = z$, $x \geq 0$

or $\exists$ $\underset{m}{1\,\boxed{c}}$ with $cA \geq 0$, $cz < 0$

but not both

Either:
a system of eqs has a positive solution
or:
a linear combination of the eqs gives
$b_1x_1 + \cdots + b_nx_n = b$
where $b_i \geq 0$, $b < 0$
(certifying insolubility)

Pf.

$\left(\exists\, x \text{ with } \begin{matrix} Ax = z \\ x \geq 0 \end{matrix}\right) \iff z \in \operatorname{cone}(A)$
$= \operatorname{conv}(0) + \operatorname{cone}(A)$
$\Updownarrow$ (FL IV)

$\exists\, a$ s.t. $\begin{matrix} aA \leq 0 \\ az > 0 \end{matrix} \iff \exists\, a, \alpha$ s.t. $a\cdot 0 \leq \alpha$, $a\cdot a_j \leq 0$, $a\cdot z > \alpha$ ∎

Farkas Lemma I

Consider $\underset{d}{m\,\boxed{A}}$, $\underset{1}{m\,\boxed{z}}$.

Either $\exists$ $\underset{1}{d\,\boxed{x}}$ with $Ax \leq z$

or $\exists$ $\underset{m}{1\,\boxed{c}}$ with $c \geq 0$, $cA = 0$, $cz < 0$

but not both

Either:
a system of inequalities has a solution
or
a positive combination of the inequalities give
$0x_1 + \cdots + 0x_d \leq b$
where $b < 0$
(certifying insolubility)

Pf
II
$\Downarrow$
I
(see book)

Farkas Lemma III

Consider $\underset{d}{m\,\boxed{A}}$, $\underset{1}{m\,\boxed{z}}$, $\underset{d}{1\,\boxed{a_0}}$, z_0. Then

$(Ax \leq z \text{ implies } a_0 x \leq z_0)$
$\Updownarrow$
$\left(\begin{matrix} \exists\, \underset{m}{1\,\boxed{c}} \geq 0 \text{ with } cA = a_0,\ cz \leq z_0 \\ \text{or} \\ \exists\, \underset{m}{1\,\boxed{c}} \geq 0 \text{ with } cA = 0,\ cz < 0 \end{matrix}\right)$

If an inequality holds for a polyhedron,
either
- it is a positive combin. of the ineqs of the poly
or
- $0x_1 + \cdots + 0x_d \leq -1$ is a pos. combin of them (i.e., polyhedron is empty)

Pf
I, II
$\Downarrow$
III
(see book)