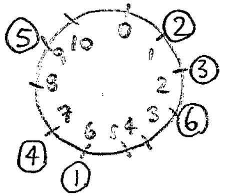
Shi arrangement:

lec 44
Dec 13

$\mathcal{S}_n$: $x_i - x_j = 0, 1$ $\quad 1 \le i < j \le n$

$\chi_{\mathcal{S}_n}(q) = \#$ of $(x_1, \ldots, x_n) \in \mathbb{F}_q^n$: $x_i \ne x_j$
$x_i \ne x_j + 1$ for $i < j$

Ex: $q = 11$
$n = 6$
$x = (6, 1, 3, 7, 9, 3)$



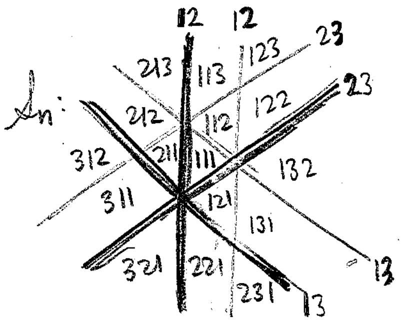
Place numbers $1, 2, \ldots, n$ around a circle, so consecutive blocks are increasing clockwise. Encoding:

Blocks: (start with block of 1) $\quad$ 14 | 5 | ∅ | 236 | ∅ | $\quad$ Starting point: 6

$n-1$ numbers in $q-n$ blocks $\quad$ q position.

$\Rightarrow \boxed{\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}} \Rightarrow \boxed{\begin{matrix} r(\mathcal{S}_n) = (n+1)^{n-1} \\ b(\mathcal{S}_n) = (n-1)^{n-1} \end{matrix}}$ parking functions!

$\mathcal{S}_n$:



Label regions:

- $x_1 < x_2 < \cdots < x_n < x_1 + 1 < \cdots < x_n + 1 \to 111111$
- cross $x_i = x_j \to$ increase i-th coord
- cross $x_i = x_j + 1 \to$ increase j-th coord

Exercise: bounded regions

Prop (Rincón): F face $\Rightarrow$ regions containing F form an interval in the poset of parking functions, under componentwise $\le$

Question: Which intervals occur?

This works for the other finite reflection groups as well.

Each one has a set of positive roots:

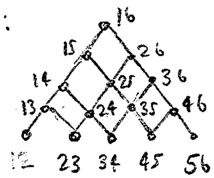
$(A_{n-1})_+ = \{e_i - e_j : 1 \le i < j \le n\}$
$(B_n)_+ = \{e_i \pm e_j : 1 \le i < j \le n\} \cup \{e_i : 1 \le i \le n\}$
$(C_n)_+ = \{e_i \pm e_j : 1 \le i < j \le n\} \cup \{2e_i : 1 \le i \le n\}$
$(D_n)_+ \ \{e_i \pm e_j : 1 \le i < j \le n\}$

Also $\Phi = \Phi_+ \cup (-\Phi_+)$ is called a <u>root system</u>

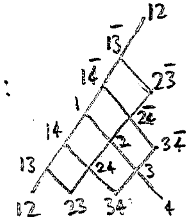
The <u>root poset</u> on Φ_+ is:

$\beta \le \gamma \iff \gamma - \beta$ is a non-neg. combin of other pos. roots.

A_3:



B_4:



$ij: e_i - e_j$
$i: 2e_i$
$\bar{ij}: e_i + e_j$

A - order ideal of P: $x < y, y \in A \Rightarrow x \in A$.

Lots of nice mathematics! The beginning!

Def / Thm The Φ-Catalan number $Cat(\Phi)$ is:

$$\left(\begin{matrix}\text{The number of regions of} \\ \text{the } \Phi\text{-Coxeter arrangement} \\ c \cdot x = -1, 0, 1 \quad c \in \Phi \\ \text{in the "positive chamber"} \\ c \cdot x \ge 0 \quad c \in \Phi\end{matrix}\right) = \left(\begin{matrix}\text{The number of} \\ \text{order ideals} \\ \text{of the root} \\ \text{poset } \Phi^+\end{matrix}\right)$$

Thm $Cat(\Phi) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}$

- e_i = "exponents" $\quad \chi_\Phi(q) = \prod (x - e_i)$
- h = "Coxeter number" $= \dfrac{\text{number of roots in } \Phi}{\text{rank of } \Phi}$

Thm $r(\Phi\text{-Shi arrangement}) = (h+1)^r$