Gian-Carlo Rota proposed this as a method of attack towards the Four Color Theorem.

Some nice facts about the roots of $\chi_G(q)$:

- real roots are ≥ 0
- if G is planar, all real roots are ≤ 5

} real

- $|\text{roots}| \leq 8$ (# vertices)
- roots of polynomials $\chi_G(q)$ are dense in $\mathbb{C}$

} complex

Thm (Stanley 1973)

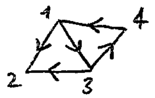
$(-1)^{\#V}\chi_G(-1) =$ # of acyclic orientations of G $=$ # of ways to orient edges of G forming no cycles.

Pf

The LHS equals $r(\mathcal{A}_G)$. Let's count regions.

A region is specified by saying, for each i—j, whether $x_i < x_j$ or $x_i > x_j$. Orient $i \leftarrow j$, $i \rightarrow j$ respectively.

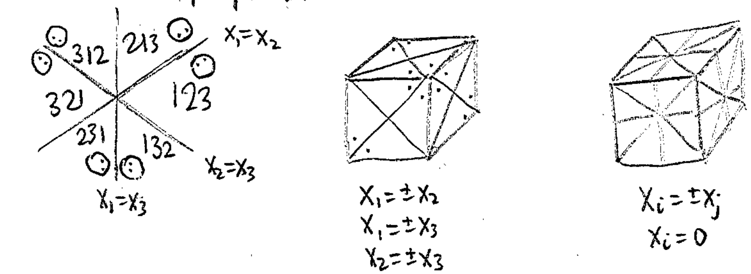
Ex: $x_1 > x_2$, $x_1 > x_3$, $x_1 < x_4$, $x_2 < x_3$, $x_3 > x_4$ $\rightarrow$



Is this a region? No because it requires $x_1 > \underset{\substack{\uparrow \\ \text{cycle in } G}}{x_3} > x_4 > x_1$

An orientation specifies a region $\iff$ its inequalities provide no contradictions $\iff$ it has no cycles. ∎

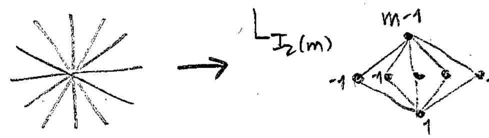
A hyperplane arrangement defines a reflection group, generated by the reflections across the hyperplanes.



The finite reflection groups

- Dihedral groups
- A_n
- B_n/C_n
- D_n
- E_6, E_7, E_8

- Dihedral group $I_2(m)$:



$\rightarrow \chi_{I_2(m)}(q) = q^2 - mq + (m-1)$

$\rightarrow r(I_2(m)) = 2m$

- A_n: $x_i = x_j$ $\quad 1 \leq i < j \leq n$

$\rightarrow \chi_{A_n}(q) = \#\{(\underset{q}{x_1}, \underset{q-1}{\ldots}, \underset{q-(n-1)}{x_n}) \in \mathbb{F}_q^n : x_i \neq x_j\} = q(q-1)\cdots(q-n+1)$

$\rightarrow r(A_n) = |\chi_{A_n}(-1)| = 1 \cdot 2 \cdots n = n!$

- B_n/C_n: $x_i = x_j$ $\quad 1 \leq i < j \leq n$, $\quad x_i = 0$ $\quad 1 \leq i \leq n$

$\rightarrow \chi_{B_n}(q) = \#\{(\underset{q-1}{x_1}, \underset{q-3}{\ldots}, \underset{q-(2n-1)}{x_n}) \in \mathbb{F}_q^n : x_i \neq \pm x_j, x_i \neq 0\} = (q-1)(q-3)\cdots(q-2n+1)$

$\rightarrow r(B_n) = |\chi_{B_n}(-1)| = 2 \cdot 4 \cdots 2n = 2^n \cdot n!$