Exs

(1)  $\chi_A(q) = q^2 - 4q + 5$ $r(A) = 10$, $b(A) = 2$

(2) $\mathcal{H}_n$: $x_1 = 0, \ldots, x_n = 0$

from defs:  $L_{\mathcal{H}_n} = B_n$

$$\chi_{\mathcal{H}_n}(q) = q^n - \binom{n}{1} q^{n-1} + \cdots + (-1)^n \binom{n}{n} q^0 = (q-1)^n$$

finite field method:

$$\chi_{\mathcal{H}_n}(q) = \#\text{ points in } \mathbb{F}_q^n \text{ on no hyp.}$$
$$= \#\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : x_i \neq 0 \text{ all } i\}$$
$$= (q-1)^n$$

$\Rightarrow r(\mathcal{H}_n) = 2^n$ $\leftrightarrow$ sign sequences of length n

$b(\mathcal{H}_n) = 0$

(3) $\mathcal{B}_n$: $x_i = x_j$

from defs: tricky!

μ is computable, but tricky

finite field method:

$$\chi_{\mathcal{B}_n}(q) = \#\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : x_i \neq x_j \text{ all } i,j\}$$
$$= \underset{\text{not } x_1}{q(q-1)}\underset{\text{not } x_1, x_2}{(q-2)} \cdots (q-n+1)$$

$\Rightarrow r(\mathcal{B}_n) = n!$ $\leftrightarrow$ perms of $[n]$

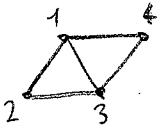
Graphical Arrangements

G – graph on vertex set $[n]$

Graphical arrangement of G:

$$\mathcal{A}_G: \quad x_i = x_j \quad \text{for} \quad i \text{—} j \text{ in } G$$

Ex

- $G =$  $\longrightarrow$ $x_1 = x_2$, $x_1 = x_3$, $x_1 = x_4$, $x_2 = x_3$, $x_3 = x_4$
- $G = K_n \longrightarrow$ braid arrangement

What is $\chi_{\mathcal{A}_G}(q)$?

From the Möbius function, hard to say.

From the finite field method,

$$\chi_{\mathcal{A}_G}(q) = \#\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \text{—} j \in G\}$$

= # of colorings of the vertices of G so that no two neighbors have the same color.

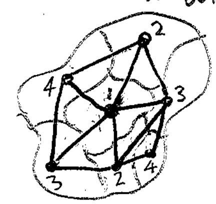
Thm. $\chi_{\mathcal{A}_G}(q)$ = # of "proper" q-colorings of G = "chromatic polynomial" of $G = \chi_G(q)$

Cor. This is a polynomial!

In those terms:

"Four-color theorem" (Guthrie, 1852)

Theorem (Appel, Haken 1976) If G is a planar graph, $\chi_{\mathcal{A}_G}(4) \neq 0$



Pf. 128 pgs of hard graph theory → 1936 "key graphs" → computer. ∎