- Q is a V-polyhedron:

Lecture 4
Sep 1, 2010

$$Q = \left\{ \binom{x}{z} \in \mathbb{R}^{d+n} : Ax \le z \right\}$$

$$= \left\{ \binom{x}{Ax+y} : x \in \mathbb{R}^d,\ y \in \mathbb{R}^n_{\ge 0} \right\}$$

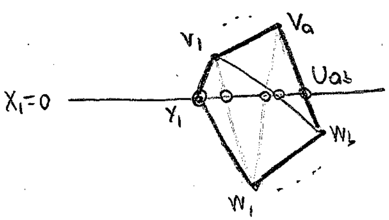
$e_1, \ldots, e_d$ unit in $\mathbb{R}^d$
$f_1, \ldots, f_n$ unit in $\mathbb{R}^n$

$$= \operatorname{cone}\left\{ \binom{e_1}{Ae_1}, \binom{-e_1}{-Ae_1}, \ldots, \binom{e_d}{Ae_d}, \binom{-e_d}{-Ae_d}, \binom{0}{f_1}, \ldots, \binom{0}{f_n} \right\}$$

- (V-polyhedron) $\cap$ (affine plane) = V-polyhedron

We sketch the proof for:

(V-polytope) $\cap$ (hyperplane $x_1 = 0$)



Let $P = \operatorname{conv}(V)$

Say $v_1, \ldots, v_a$ have $x_1 > 0$

$w_1, \ldots, w_b$ have $x_1 < 0$

$y_1, \ldots, y_c$ have $y_1 = 0$

Let $U_{rs} = \overline{V_r W_s} \cap (X_1 = 0) = \dfrac{(V_r)_1}{(V_r)_1 - (U_s)_1} U_s + \dfrac{-(U_s)_1}{(V_r)_1 - (U_s)_1} V_r$

for $r = 1, \ldots, a \quad s = 1, \ldots, b$.

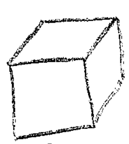
Then

$$P \cap (x_1 = 0) = \operatorname{conv}\{(Y_i)_{1 \le i \le c}, (U_{rs})_{\substack{1 \le r \le a \\ 1 \le s \le b}}\}$$

$\supseteq$: clear

$\subseteq$: ugly computation. (Exercise) ■

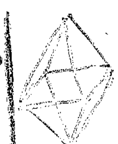
Note



C_3

conv (111, 11-1, 1-11, 1-1-1, -111, -11-1, -1-11, -1-1-1)

$$\begin{bmatrix} 1&0&0 \\ -1&0&0 \\ 0&1&0 \\ 0&-1&0 \\ 0&0&1 \\ 0&0&-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



$\Diamond_3$

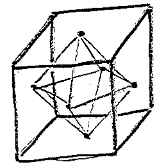
conv (100, 010, 001, -100, 0-10, 00-1)

$$\begin{bmatrix} 1&1&1 \\ 1&1&-1 \\ 1&-1&1 \\ 1&-1&-1 \\ -1&1&1 \\ -1&1&-1 \\ -1&-1&1 \\ -1&-1&-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

P polytope $\longrightarrow P^\Delta = \{ a \in \mathbb{R}^d : a \cdot x \le 1 \text{ for all } x \in P \}$

Idea: V for $P \leftrightarrow$ H for P^Δ

H for $P \leftrightarrow$ V for P^Δ (will show)



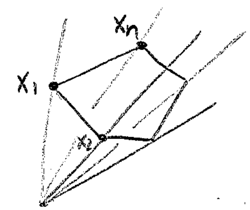
There is an important symmetry H $\longleftrightarrow$ V
We will discuss it in detail later.

Carathéodory's Theorem

(i) If $x \in \operatorname{cone}(X)$
then $x \in \operatorname{cone}(Y)$ for some $Y \subseteq X$, $|Y| \le \dim(\operatorname{cone} X)$

(ii) If $x \in \operatorname{conv}(X)$
then $x \in \operatorname{conv}(Y)$ for some $Y \subseteq X$, $|Y| \le \dim(\operatorname{conv} X) + 1$

Pf



(i) Write x as a pos. comb. of X:

$$x = t_1 x_{i_1} + \cdots + t_k x_{i_k} \qquad t_i > 0$$

with k minimum.

- If $k \le d$, done
- If $k > d$, then $t_1 x_{i_1}, \ldots, t_k x_{i_k}$ are lin. dep., say

$$\lambda_1 (t_1 x_{i_1}) + \cdots + \lambda_k (t_k x_{i_k}) = 0$$

Assume WLOG that $\lambda_1 > 0$, and λ_1 is the largest λ_i.

Then $\quad t_1 x_{i_1} = -t_2 \dfrac{\lambda_2}{\lambda_1} x_{i_2} - \cdots - t_k \dfrac{\lambda_k}{\lambda_1} x_{i_k}$

so

$$x = \underbrace{t_2 (1 - \lambda_2/\lambda_1)}_{\ge 0} x_{i_2} + \cdots + \underbrace{t_k (1 - \lambda_k/\lambda_1)}_{\ge 0} x_{i_k}$$

is a smaller expression for x.

(ii) Follows from $\quad x \in \operatorname{conv}(X) \iff \binom{1}{x} \in \operatorname{cone}\binom{1}{X}$ ■