B $P = B_n$

$1,2,\dots,n$: properties that elements of a set E can have.

E_i: elements of E with property i.

$$\boxed{\begin{pmatrix}\#\text{ of elts with}\\ \text{none of the}\\ \text{properties}\end{pmatrix} = \left|E - (E_1 \cup \dots \cup E_n)\right| = |E| - \sum_i |E_i| + \sum_{i<j} |E_i \cap E_j| - \dots + (-1)^n |E_1 \cap \dots \cap E_n|}$$

"Inclusion-Exclusion Formula"

C P = poset of faces of a polytope

L_F = Ehrhart poly of $F \leq P$

L_{F^o} = rel. int. Ehr. poly of $F \leq P$

Then $L_F(t) = \sum_{G \leq F} L_{G^o}(t)$

$$\Rightarrow L_{F^o}(t) = \sum_{G \leq F} \mu(G,F) L_G(t)$$

$$(-1)^{\dim F} L_F(-t) = \sum_{G \leq F} (-1)^{\dim F - \dim G} L_G(t)$$

$$\boxed{L_F(-t) = \sum_{G \leq F} (-1)^{\dim G} L_G(t)}$$

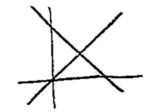
An alternative approach to Ehrhart reciprocity.

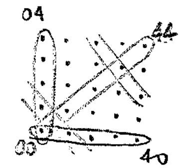
D P = intersection poset of arrangement $\mathcal{A}$

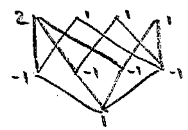
What can we count in $\mathbb{R}^n$?

What if we try $\mathbb{F}_q^n$?

Ex $\mathcal{A}$: $x=0$, $y=0$, $x=y$, $x+y=1$

In $\mathbb{R}^2$: 

In $\mathbb{F}_q^2$: 

$L_{\mathcal{A}}$: 

$\chi_{\mathcal{A}}(q) = q^2 - 4q + 5$

$$\boxed{\begin{array}{l}\textbf{Theorem } \mathcal{A}\text{-arrangement in } \mathbb{F}_q^n \\ \chi_{\mathcal{A}}(q) = \#\text{ of points in } \mathbb{F}_q^n \text{ on no hyperplane of } \mathcal{A} = |\mathbb{F}_q^n \setminus \mathcal{A}|\end{array}}$$

Pf $g(F)$ = # of points on F $= q^{\dim F}$

$f(F)$ = # of points in F, not in any $G \subsetneq F$

Then $g(F) = \sum_{G \geq F} f(G)$

so $f(F) = \sum_{G \geq F} \mu(F,G)\, g(G) = \sum_{G \geq F} \mu(F,G)\, q^{\dim G}$

$$|\mathbb{F}_q^n \setminus \mathcal{A}| = f(\emptyset) = \sum_G \mu(G)\, q^{\dim G} = \chi_{\mathcal{A}}(q). \quad \blacksquare$$

Finite Field Method:

$\mathcal{A}$ - arrangement with integer coefficients

$\mathcal{A}_q$ - arrangement over $\mathbb{F}_q$ with same equations.

Then $\chi_{\mathcal{A}}(q) = |\mathbb{F}_q^n \setminus \mathcal{A}_q|$

for any large enough prime q

Pf Need $L_{\mathcal{A}} \cong L_{\mathcal{A}_q}$ so $\chi_{\mathcal{A}} = \chi_{\mathcal{A}_q}$. Just parameterize each flat F. If q is large enough, the same param. will work over $\mathbb{F}_q$ $\blacksquare$