Our next goal: computing characteristic polynomials

Lec 38
Nov 29

We need: Möbius inversion

<u>Def</u> The <u>two-variable Möbius function</u> of a poset P is

$$\mu : \{(x,y) \mid x \le y \text{ in } P\} \to \mathbb{Z}$$

defined by

$$\mu(x,y) = \begin{cases} 1 & \text{if } y = x \\ -\sum_{x \le z < y} \mu(x,z) & \text{if } x < y. \end{cases}$$

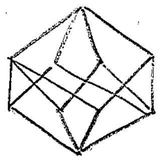
So if we let

$$P_{\ge x} = \{ y \in P \mid y \ge x \}$$

then

$$\mu_P(x,y) = \mu_{P_{\ge x}}(y)$$

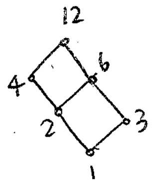
<u>Ex. 1</u>



<u>Ex. 2</u>

D_n = poset of divisors of n

We stated:
$$\mu(m) = \begin{cases} (-1)^t & \text{if } m = p_1 \cdots p_t \\ 0 & \text{otherwise.} \end{cases}$$



Then,
$$\mu_{D_n}(j,k) = \mu_{(D_n)_{\ge j}}(k) \quad \leftarrow \text{divisors of } n \text{ multiples of } j$$
$$= \mu_{D_{n/j}}(k/j) \quad \leftarrow \text{divisors of } n/j$$
$$= \begin{cases} (-1)^t & \text{if } k/j = p_1 \cdots p_t \\ 0 & \text{otherwise.} \end{cases}$$

Möbius Inversion Formula

P poset $\quad f, g : P \to \mathbb{Z}$ — or any abelian group

If
$$g(x) = \sum_{y \le x} f(y) \quad (\text{all } x)$$
then
$$f(x) = \sum_{y \le x} \mu(y,x)\, g(y) \quad (\text{all } x)$$

Also:
$$g(x) = \sum_{y \ge x} f(y) \iff f(x) = \sum_{y \ge x} \mu(x,y)\, g(y)$$

<u>Pf</u> For fixed x,
$$\sum_{y \le x} \mu(y,x)\, g(y) = \sum_{y \le x} \Big[\mu(y,x) \sum_{z \le y} f(z) \Big] = \sum_{z \le y \le x} \mu(y,x) f(z).$$

The coeff of $f(z)$ is
$$\sum_{z \le y \le x} \mu(y,x) = \begin{cases} 1 & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$
thanks to the lemma: ∎

We know:
$$\sum_{a \le t \le b} \mu(a,t) = \begin{cases} 1 & a = b \\ 0 & a < b \end{cases}$$

<u>Lemma:</u>
$$\sum_{a \le t \le b} \mu(t,b) = \begin{cases} 1 & a = b \\ 0 & a < b \end{cases}$$

<u>Pf</u> Fix a, "poset induct" on b

- $b = a$ ✓
- Sup true for all c : $a \le c < b$. Prove for b

$$\sum_{t : a \le t \le b} \mu(t,b) = \sum_{t : a \le t < b} \Big(-\sum_{u : t \le u < b} \mu(t,u) \Big) + \mu(b,b)$$
$$= -\sum_{u : a \le u < b} \underbrace{\Big(\sum_{t : a \le t \le u} \mu(t,u) \Big)}_{\substack{1 \text{ if } a = u \\ 0 \text{ if } a < u}} + 1$$
$$= -1 + 1 = 0. \ ∎$$

Applications:

(A) $P = D_n$ (divisors of $n = p_1^{a_1} \cdots p_t^{a_t}$)

$g(d)$ = # of $1 \le k \le n$ such that $d \mid k$ $= n/d$

$f(d)$ = # of $1 \le k \le n$ such that $(n,k) = d$ $= ?$

Note:
$$g(d) = \sum_{d \mid m \mid n} f(m) \Rightarrow f(d) = \sum_{d \mid m \mid n} \mu(d,m)\, g(m) = \sum_{d \mid m} \mu(m/d) \cdot n/m$$

$\varphi(n)$ = # of $1 \le k \le n$ coprime with n $\Rightarrow \varphi(n) = f(1) = n - \sum n/p_i + \sum n/p_i p_j - \ldots$

$$\boxed{\varphi(n) = n\left(1 - \tfrac{1}{p_1}\right) \cdots \left(1 - \tfrac{1}{p_t}\right)}$$