Lemma For any flat F, $\mu(F) = \underbrace{\sum_{\substack{B\subseteq A \\ \cap B = F}} (-1)^{|B|}}_{\nu(F)}$

Pf. Check $\sum_{G\le F} \nu(F) = 0$

$$\sum_{G\le F}\sum_{\substack{B\subseteq A \\ \cap B=G}}(-1)^{|B|} = \sum_{\substack{B\subseteq A \\ \cap B\supseteq F}}(-1)^{|B|} = \sum_{\substack{B\subseteq A \\ B\supseteq F \\ \text{for } B\in \mathcal{B}}} (-1)^{|B|}$$

$$= \sum_{B\subseteq \mathcal{F}} (-1)^{|B|} = (1+(-1))^{|\mathcal{F}|} = 0. \qquad \mathcal{F} = \{A\in A : A\supseteq F\}$$ ∎

Prop (Whitney) $\chi_A(q) = \sum_{B\subseteq A}(-1)^{|B|} q^{\dim(\cap B)}$

Pf $$\chi_A(q) = \sum_{F\in L_A}\mu(F)\, q^{\dim F}$$
$$= \sum_{F\in L_A}\Bigg(\sum_{\substack{B\subseteq A \\ \cap B=F}}(-1)^{|B|}\Bigg) q^{\dim F} = RHS$$ ∎

Thm (Deletion-Contraction)
A hyp arr
$H \in A$
$$\Rightarrow \chi_A(q) = \chi_{A\setminus H}(q) - \chi_{A/H}(q)$$

Pf
$$\chi_A(q) = \sum_{\substack{B\subseteq A \\ H\notin B}}(-1)^{|B|}q^{\dim(\cap B)} + \sum_{\substack{B\subseteq A \\ H\in B}}(-1)^{|B|}q^{\dim(\cap B)}$$
$$= \sum_{C\subseteq A\setminus H}(-1)^{|C|}q^{\dim(\cap B)} + \sum_{C\subseteq A\setminus H}(-1)^{|C\cup H|} q^{\dim(\cap C\cap H)}$$

First Term: $\chi_{A\setminus H}(q)$

Second Term: $$-\sum_{C\subseteq A\setminus H}(-1)^{|C|} q^{\dim\left(\bigcap_{C\in \mathcal{C}}(C\cap H)\right)}$$

$$= -\sum_{C'\subseteq A/H} (-1)^{|C'|} q^{\dim \cap C'} = \chi_{A/H}(q)$$

where $C' = \{C\cap H : C\in C\}$ ∎

Then we have

Proof of Zaslavsky's Theorem

Induct on $|A|$

Do it for $r(A)$. (Similar for $b(A)$.)

- $|A|=0$: $r(A)=1$, $\chi_A(q)=q^d$, $r(A) = (-1)^d\chi_A(-1)$ ✓
- $|A|>0$:

$$(-1)^d\chi_A(-1) = (-1)^d\chi_{A\setminus H}(-1) - (-1)^d\chi_{A/H}(-1)$$
$$= r(A\setminus H) + r(A)$$
$$= r(A).$$ ✓ ∎

So: to compute $r(A), b(A)$ we just need to compute $\chi_A(q)$.
How do we do that?
We'll see.