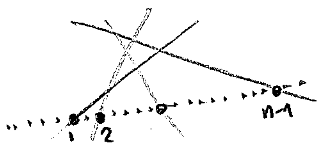
Ex: G_n: n lines in <u>general position</u> in $\mathbb{R}^2$

└ no 2 parallel
no 3 concurrent

G_{n-1}
$\downarrow$
G_n



$r(G_n) = \#(\text{old regions}) + \#(\text{old regions cut in two by last line})$

$r(G_n) = r(G_{n-1}) + n$

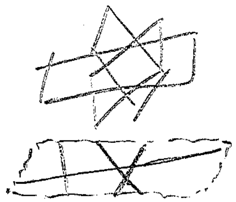
$b(G_n) = b(G_{n-1}) + (n-2)$

$\Rightarrow \boxed{r(G_n) = \binom{n}{2} + n + 1}$ $\boxed{b(G_n) = \binom{n}{2} - n + 1}$

Ex H_n: n planes in <u>general position</u> in $\mathbb{R}^3$

└ no 2 parallel
no 3 $\cap$ on a plane
no 4 $\cap$

H_{n-1}
$\downarrow$
H_n



$r(H_n) = \#(\text{old regions}) + \#(\text{old regions cut in two})$

$= r(H_{n-1}) + r(G_{n-1})$

$r(H_n) = r(H_{n-1}) + \binom{n}{2} + n + 1$

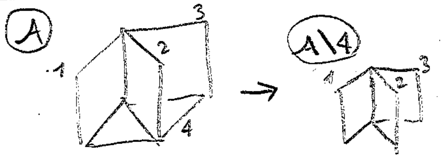
$b(H_n) = b(H_{n-1}) + \binom{n}{2} - n + 1$

$\Rightarrow \boxed{r(G_n) = \binom{n}{3} + \binom{n}{2} + n + 1}$ $\boxed{b(G_n) = \binom{n}{3} - \binom{n}{2} + n - 1}$

This suggests:

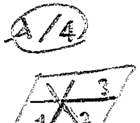
$\mathcal{A} = \{H_1, \dots, H_n\}$ in $\mathbb{R}^d$

$H_1 \in \mathcal{A}$



Define:

- deletion:
  $\mathcal{A} \setminus H_1 = \{H_2, \dots, H_n\}$ in $\mathbb{R}^d$
- contraction:
  $\mathcal{A}/H_1 = \{H_2 \cap H_1, \dots, H_n \cap H_1\}$ in $H_1 \cong \mathbb{R}^{d-1}$



Def For $\mathcal{B} \subseteq \mathcal{A}$ let $\operatorname{rank}(\mathcal{B}) = d - \dim(\cap \mathcal{B})$

Lemma $r(\mathcal{A}) = r(\mathcal{A} \setminus H) + r(\mathcal{A}/H)$

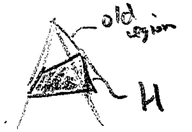
$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}\setminus H) + b(\mathcal{A}/H) & \text{if } \operatorname{rank}(\mathcal{A}\setminus H) = \operatorname{rank} \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$

Pf $\mathcal{A} \setminus H$ has $r(\mathcal{A}\setminus H)$ regions.
Now add H.

$r(\mathcal{A}) = \#(\text{old regions}) + \#(\text{old regions cut in two})$

$= r(\mathcal{A}\setminus H) + r(\mathcal{A}/H)$

Similar for $b(\mathcal{A})$. ∎



Def Say $\mathcal{A}$ is in <u>general position</u> if, for all $\mathcal{B} \subseteq \mathcal{A}$,

$$\operatorname{rank}(\mathcal{B}) = \begin{cases} |\mathcal{B}| & \text{if } |\mathcal{B}| \le d \\ d & \text{otherwise} \end{cases}$$

Prop If $\mathcal{A}_n^d = \{H_1, \dots, H_n\}$ in $\mathbb{R}^d$ is in general position, then

$$r(\mathcal{A}_n^d) = \binom{n}{d} + \binom{n}{d-1} + \dots + \binom{n}{1} + \binom{n}{0}$$

$$b(\mathcal{A}_n^d) = \binom{n}{d} - \binom{n}{d-1} + \dots + (-1)^{d-1}\binom{n}{1} + (-1)^d \binom{n}{0}$$

Pf Induct on $n+d$.

$r(\mathcal{A}_n^d) = r(\mathcal{A}_n^d \setminus H_1) + r(\mathcal{A}_n^d / H)$ $\qquad b(\mathcal{A}_n^d) = \dots$ (same)

$= r(\mathcal{A}_{n-1}^d) + r(\mathcal{A}_{n-1}^{d-1})$

$= \binom{n-1}{d} + \binom{n-1}{d-1} + \dots + \binom{n-1}{1} + \binom{n-1}{0}$

$+ \binom{n-1}{d-1} + \binom{n-1}{d-2} + \dots + \binom{n-1}{0}$

$= \binom{n}{d} + \binom{n}{d-1} + \dots + \binom{n}{1} + \binom{n}{0}$ ∎

In particular: (reality check)

$\mathcal{H}_n$: $x_i = 0$ $\quad 1 \le i \le n$ $\rightarrow r(\mathcal{H}_n) = \binom{n}{n} + \binom{n}{n-1} + \dots + \binom{n}{0} = 2^n$ ✓