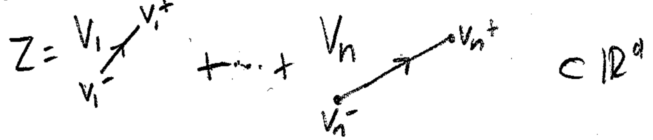
In general:



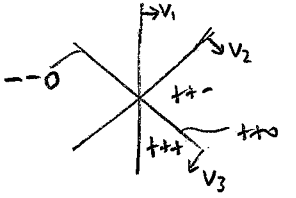
Let $c \in (\mathbb{R}^d)^*$, What is c-max face Z_c?

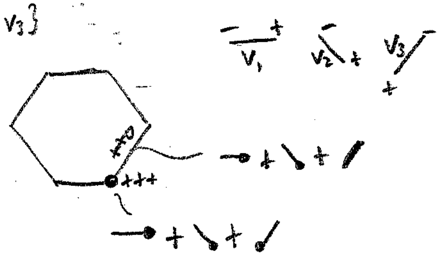
$$Z_c = \underbrace{(V_1)_c}_{} + \cdots + (V_n)_c$$

$$= \begin{cases} V_1^+ & \text{if } c\cdot V_1 > 0 \\ V_1 & \text{if } c\cdot V_1 = 0 \\ V_1^- & \text{if } c\cdot V_1 < 0 \end{cases}$$

So $\begin{pmatrix} c, c' \text{ in the} \\ \text{same cone} \\ \text{of } \mathcal{N}(Z) \end{pmatrix} \Leftrightarrow \begin{pmatrix} c\cdot V_i,\ c'\cdot V_i \\ \text{have the} \\ \text{same sign} \\ (\text{all } i) \end{pmatrix} \Leftrightarrow \begin{pmatrix} c, c' \text{ in the} \\ \text{same face} \\ \text{of } \mathcal{A} \end{pmatrix}$ ∎

Sign vectors: Let $V=\{V_1,V_2,V_3\}$





These are called "signed covectors of V"
We have a bijection:

$$\begin{pmatrix} \text{signed} \\ \text{covectors} \\ \text{of } V \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{faces} \\ \text{of } \mathcal{A}_V \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Nonempty} \\ \text{faces of } Z(V) \end{pmatrix}$$

So we would do well understanding arrangements and their faces.

Note: not all sign vectors are realized as faces,

e.g, + − +

HYPERPLANE ARRANGEMENTS

A hyp arr is $\mathcal{A}=\{H_1,\dots,H_n\}$ in $\mathbb{R}^d$ with

$H_i = \{x : a_i\cdot x = b_i\}$

If all $b_i=0$, call it central.

"Faces": as above

"Regions": d-dim faces ⟺ conn comps of $\mathbb{R}^d - \bigcup_{i=1}^n H_i$

$r(\mathcal{A})$ = # regions

$b(\mathcal{A})$ = # (relatively bounded) regions

Ex:

$r(\mathcal{A})=11$
$b(\mathcal{A})=3$

Ex. "Braid arrangement"

A_{n-1}: $x_i = x_j$ $1\le i<j\le n$ in $\mathbb{R}^n$

To specify a region R, I need to decide, for each $i<j$ whether $\begin{matrix} c_i<c_j \\ c_i>c_j \end{matrix}$ for $c\in R$

So

$$\begin{pmatrix} \text{regions} \\ \text{of } A_{n-1} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{perms} \\ \text{of } [n] \end{pmatrix}$$

So $r(A_{n-1}) = n!$, $b(A_{n-1}) = 0$.

No surprise because $V=\{e_i-e_j \quad 1\le i<j\le n\}$

$$\begin{tikzcd} & V=\{e_i-e_j\} \arrow[dl] \arrow[dr] & \\ A_V = A_{n-1} & & Z(V) = \Pi_{n-1} \end{tikzcd}$$

- regions of A_{n-1} ⟷ vertices of Π_{n-1} ⟷ perms of S_n
- faces of Π_{n-1} ⟷ faces of A_{n-1} ⟷ ordered set partitions of [n]