A more explicit explanation for $\mathrm{Vol}\,\Pi_n = n^{n-2}$:

lec 33
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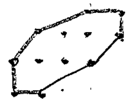
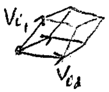
## Zonotopes

**Def** A <u>zonotope</u> is a Minkowski sum of segments.

**Ex**

$| + \_ + / + / =$ 

**Prop** The zonotope $Z = v_1 + \cdots + v_n \subset \mathbb{R}^d$ can be tiled into parallelepipeds:

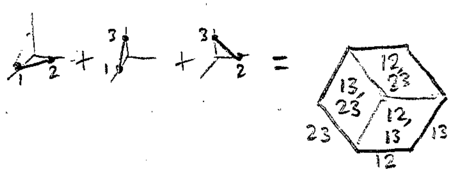
$v_{i_1} + \cdots + v_{i_d} =$ 



one for each basis $\{v_{i_1} \cdots v_{i_d}\}$ of $\mathbb{R}^d$.

**Cor**

$$\mathrm{Vol}\, Z = \sum_{\substack{\{v_{i_1}, \dots, v_{i_d}\} \\ \text{basis}}} |\det(v_{i_1}, \dots, v_{i_d})|$$

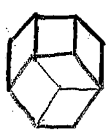
**Ex** $\Pi_3 =$ 



- one parallelepiped per spanning tree
- all have volume 1

This is true for $\Pi_n$ as well.

<u>Sketch of Proof</u>: "Just induct":



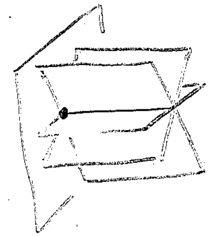
But to make this rigorous we need some machinery. $\Rightarrow$ hyperplane arrangements, matroids,

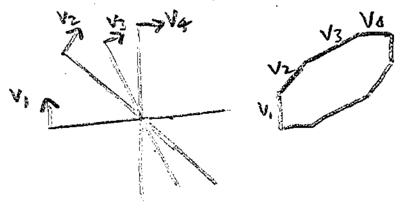


The "dual" object can be easier to work with:

A <u>hyperplane arrangement</u> $\mathcal{A} = \{H_1, \dots, H_n\}$ in $\mathbb{R}^d$ is a collection of hyperplanes

$$H_i = \{ c \in (\mathbb{R}^d)^* : \ c \cdot v_i = 0 \}$$

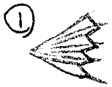
Ex: 

Ex: 



hyp. arr. $\longleftrightarrow$ zonotope

**Def** A fan $\mathcal{F} = \{C_1, \dots, C_N\}$ in $\mathbb{R}^d$ is a polyhedral complex of cones $C_i$. It is <u>complete</u> if $\bigcup_i C_i = \mathbb{R}^d$ and pointed if $\{0\} \in \mathcal{F}$.

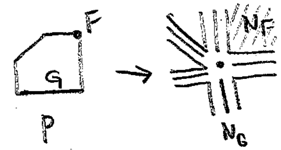
**Ex** ① 

② $\mathcal{A}$ hyp. arr.
Faces: $c \in \mathbb{R}^d$: $c \cdot v_i \gtreqless 0$ (choose for each $i$)
$\mathcal{F} = \{\text{faces of } \mathcal{A}\}$ 

③ $P$ polytope

For each face $F$,

$N_F = \{ c \in (\mathbb{R}^d)^* : \ P_c \supset F \}$
$=$ directions where $F$ is max

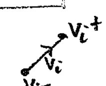




<u>Normal fan of $P$</u>:

$\mathcal{N}(P) = \{N_F : F \text{ face of } P\}$

**Prop** The normal fan of the zonotope $Z = v_1 + \cdots + v_n$ is the fan of the arrangement $H_i : v_i \cdot x = 0$

Pf. (For vertices) $v$ vertex of $Z \rightarrow v = v_1^+ + v_2^- + v_3^+ + v_4^+$ 



$\begin{pmatrix} c \text{ max in} \\ Z = v_1 + v_2 + v_3 + v_4 \\ \text{at } v \end{pmatrix} \Leftrightarrow \begin{pmatrix} c \text{ max in } v_1 \text{ at } v_1^+, \text{ in } v_2 \text{ at } v_2^- \dots \end{pmatrix} \Leftrightarrow \begin{pmatrix} c \cdot v_1 > 0 \\ c \cdot v_2 < 0 \\ \vdots \end{pmatrix} \rightsquigarrow$ face of $\mathcal{A}$ ∎