Lemma $(a_1,\dots,a_n)$ is a parking function $\Leftrightarrow$ all cars can park

Pf $\Leftarrow$ If everyone can park:

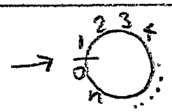
The cars with pref spot $\le i$ can only park at $1,2,\dots,i$ $\Rightarrow$ there's $\le i$ of them

$\Rightarrow$ Sup a car can't park

Say $1,2,\dots,k$ are taken, $k+1$ is not.

Then the cars at $1,2,\dots,k$ and the new car have prefs $\le k$. ☒

Thm There are $(n+1)^{n-1}$ parking functions of length n

Change the road:  and let a car circle around until it can park. Allow 0 as a preference

$\Rightarrow$ Now there are $(n+1)^n$ preferences possible

If $(a_1,\dots,a_n)$ leaves spot k empty

$(a_1+1,\dots,a_n+1)$ " " $k+1$ " (mod $n+1$)

⋮

$(a_1+n,\dots,a_n+n)$ " " $k+n$ empty

So exactly one of these leave spot 0 empty

$\Rightarrow$ exactly one of them is a parking function (and hence has no 0s in the preferences)

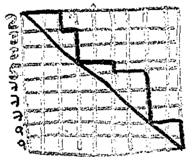
So # of parking functions $= \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$ ☒

By the way…

PF_3: 123, 132, 213, 231, 312, 321 | 223, 232, 322 | 133, 313, 331 | 233, 323, 332 | 333

How many equivalence classes of parking functions?

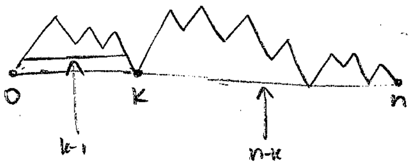
Equivalently, how many sequences $1 \le j_1 \le j_2 \le \dots \le j_n \le n$ satisfy $j_i \ge i$ (all i)?

23357799 →  "Dyck path"
- steps → and ↓
- stays above diagonal

Let d_n be the number of Dyck paths of length n.

$d_n = \sum_{k=1}^{n}$ number of Dyck paths returning to the diagonal for the first time at k

$= \sum_{k=1}^{n} d_{k-1} d_{n-k}$



Also $d_1 = 1$

So $d_n = C_n = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers!

Ok, while we are at it...

Thm (Stasheff-Postnikov) The Minkowski sum $\sum_{i<j} \Delta_{\{i,\dots,j\}}$ is a realization of the associahedron.

Recall: The associahedron is a polytope with

C_n vertices $\leftrightarrow$ triangulations of $(n+2)$-gon

faces $\leftrightarrow$ subdivisions of $(n+2)$-gon