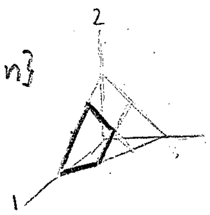
lec31
Nov5

Another nice example:

The Pitman-Stanley polytope:

$$PS_n = \{ y \in \mathbb{R}^n : y_i \ge 0,\ y_1+\dots+y_i \ge i,\ y_1+\dots+y_n = n \}$$



Let $\Delta_I = \mathrm{conv}\{e_i : i \in I\}$

Lemma

$PS_n = \Delta_{\{1\}} + \Delta_{\{1,2\}} + \dots + \Delta_{\{1,2,\dots,n\}}$

$\Delta_{[i]} = \Delta_{\{1,\dots,i\}}$

Pf

$\supseteq$: In
$\Delta_{\{1\}}$: $y_1 \ge 1$, $y_1+y_2 \ge 1$, ..., $y_1+\dots+y_n = 1$
$\Delta_{\{1,2\}}$: $y_1 \ge 0$, $y_1+y_2 \ge 1$, ..., $y_1+\dots+y_n = 1$
$\vdots$
$\Delta_{\{1,2,\dots,n\}}$: $y_1 \ge 0$, $y_1+y_2 \ge 0$, ..., $y_1+\dots+y_n = 1$

In RHS: $y_1 \ge 1$, $y_1+y_2 \ge 2$, ..., $y_1+\dots+y_n = n$.

$\subseteq$: Exercise.

So $$\mathrm{Vol}(PS_n) = \sum_{1 \le i_j \le n} \mathrm{Vol}(\Delta_{[i_1]}, \Delta_{[i_2]}, \dots, \Delta_{[i_{n-1}]})$$

Def A reverse parking function $(a_1,\dots,a_n) \in [n]^n$ is a sequence having $\le i$ elements less than or equal to i $(1 \le i \le n)$

$$\boxed{\mathrm{Vol}(\Delta_{[i_1]}, \dots, \Delta_{[i_m]}) = \begin{cases} 1/(n-1)! & \text{if } (i_1,\dots,i_{n-1}) \text{ is a reverse parking fn} \\ 0 & \text{otherwise} \end{cases}}$$

Pf The corresp system:

$$\begin{array}{r} j_1-1 \to \\ j_2-1 \to \\ \vdots \\ \\ j_{n-1}-1 \to \end{array} \begin{bmatrix} * & * & & & \\ * & * & & & \\ * & * & * & & \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} x_2 \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix} \qquad \{i_1 \dots i_m\} = \{j_1 \le \dots \le j_m\}$$

has one sol when $\det \ne 0 \iff j_1 - 1 \ge 1 \ \cdots\ j_{n-1}-1 \ge n-1 \iff \le i$ $(i_k - 1)$'s are $\le i$

It is in $(\mathbb{C}\setminus 0)^{n-1}$ automatically, since RHS has no zeros ∎

For $n=4$:

123, 132, 213, 231, 312, 321 | 223, 232, 322 | 233, 323, 332 | 133, 313, 331 | 333

Corollary

$(n-1)!\,\mathrm{Vol}(PS_n)$ = number of parking functions of length $n-1$
$= n^{n-2}$

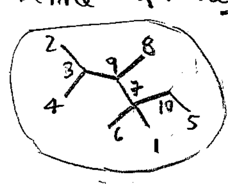
Open Question Can you tile the permutohedron Π_n with $(n-1)!$ copies of PS_n?
(maybe modulo a unimodular linear transf)

Combinatorial aside

Thm (Cayley) The number of spanning trees of K_n is n^{n-2}

Pf Define $f: \text{trees} \to [n]^{n-2}$

$T \mapsto$ Sequentially remove highest leaf, record neighbor, until two are left.



$\mapsto (9,7,10,7,3,3,9,7) = \mathrm{code}(T)$

Note: i has d neighbors $\to$ i appears $d-1$ times in code

So to recover T from code(T), sequentially look for the largest label missing from the rest of the sequence and connect it:

(9, 7, 10, 7, 3, 3, 9, 7)

8 6 5 10 4 2 3 9 7 1 ∎

Why "parking fns"?

Let $(a_1,\dots,a_n) \in [n]^n$

Sup n cars want to park on a oneway street $\longrightarrow$ n $n-1$ ... 2 1

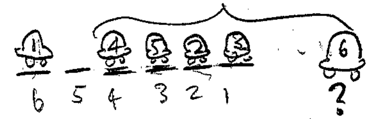
They come in order

Car i tries to park in spot a_i. If it's busy, it parks in the next available one.

Can they all park?

5 a_i's are ≤ 4

pref: (6, 2, 2, 4, 4, 3) $\longrightarrow$



car 1 2 3 4 5 6