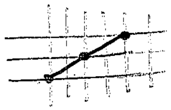
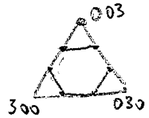
Now we use mixed volumes, but we have a problem:
Π_n is $(n-1)$-dimensional!

Aside: Relative volume.

Sup $P \subset \mathbb{R}^d$ is a lattice polytope with $\dim P < d$
let $S = \operatorname{aff} P$, affine span. We often prefer to compute volumes relative to the lattice $S \cap \mathbb{Z}^d$, so a unit cube in that lattice has vol 1



← rel. vol. = lattice length = 2



rel vol $= 6 \cdot \Delta = 6 \cdot \frac{1}{2}$

Note:
If P has Ehrhart poly $L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \dots + 1$

$c_d = \text{relvol } P$ $\quad c_{d-1} = \frac{1}{2}(\text{rel. surf. area of } P)$

So

$$\operatorname{RelVol}(\Pi_n) = \sum_{\substack{(i_1,j_1)\\ \vdots \\ (i_{n-1},j_{n-1})}} \operatorname{RelVol}(\Delta_{i_1 j_1}, \dots, \Delta_{i_{n-1} j_{n-1}})$$

let's project $e_i \mapsto e_i$ $(1 \le i \le n-1)$, $e_n \mapsto 0$ to lose a dimension.

Lemma $\operatorname{RelVol}(\Delta_{i_1 j_1}, \dots, \Delta_{i_{n-1} j_{n-1}}) = \begin{cases} 1/(n-1)! & \text{if } (i_1,j_1) \dots (i_{n-1},j_{n-1}) \text{ are a spanning tree of } K_n \\ 0 & \text{otherwise} \end{cases}$

(McKelvey, McLennan 97 / Postnikov 09)

Pf $\operatorname{RelVol}(\Delta_{i_1 j_1}, \dots, \Delta_{i_{n-1} j_{n-1}}) = \operatorname{Vol}(\Delta^*_{i_1 j_1}, \dots, \Delta^*_{i_{n-1} j_{n-1}})$ proj. to $x_n = 0$

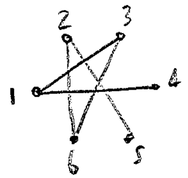
This is the # of isolated $\mathbb{C}^*$ sols to

$$\begin{cases} * x_{i_1} + * x_{j_1} = 0 \\ \vdots \\ * x_{i_{n-1}} + * x_{j_{n-1}} = 0 \end{cases}$$

Ex:

$$\begin{array}{l} \_x_1 + \_x_3 = 0 \\ \_x_2 + \quad = * \\ \_x_1 + \_x_4 = 0 \\ \_x_3 = * \\ \_x_2 + \_x_5 = 0 \end{array}$$

The corresp graph is



(0, 1, or ∞ many sols)

- If G is a spanning tree,
  - for each vertex v there is a unique path to n, which gives the value of x_v. (nonzero)
- If it is not,
  some k vertices, edges form a cycle
  - if they don't involve vertex n,

$$\begin{cases} \_x_1 + \_x_2 = 0 \\ \_x_2 + \_x_3 = 0 \\ \_x_1 + \_x_3 = 0 \end{cases} \longrightarrow \begin{array}{l} x_1 = x_2 = x_3 = 0 \\ \Rightarrow \text{not in } (\mathbb{C}^*)^n \end{array}$$

  - if they do involve it, then

$$\begin{cases} \_x_1 = * \\ \_x_1 + \_x_2 = * \\ \_x_2 + \_x_3 = * \\ \_x_3 = * \end{cases} \longrightarrow \text{inconsistent system}$$

$\square$

Therefore

Thm $\operatorname{Vol}(\Pi_n) = \#$ of spanning trees of K_n $= n^{n-2}$