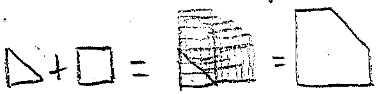
Lecture 3
Aug 30, '10

Other constructions of polytopes:

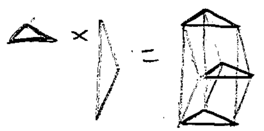
$P, Q \subset \mathbb{R}^d \longrightarrow P \cap Q \subset \mathbb{R}^d$

$\longrightarrow P + Q = \{p+q : p \in P, q \in Q\} \subset \mathbb{R}^d$



"Minkowski sum"

$P \subset \mathbb{R}^d, Q \subset \mathbb{R}^e \longrightarrow P \times Q = \left\{ \begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{R}^{d+e} : p \in P, q \in Q \right\} \subset \mathbb{R}^{d+e}$



$\subset \mathbb{R}^4$

Note: $C_d = [-1,1]^d$

These are also polytopes. (HW)

V- and H- descriptions of polytopes

We have described in terms of
- convex hulls (Vertices)
- inequalities (Halfspaces)

Main Theorem for Polytopes

$\left(\begin{array}{c}\text{convex hulls of} \\ \text{finitely many points}\end{array}\right) = \left(\begin{array}{c}\text{bounded intersections} \\ \text{of halfspaces}\end{array}\right) = \text{polytopes}$

Temporary definitions:

$\mathrm{cone}(w_1, \ldots, w_m) = \{\sum \lambda_i w_i, \lambda_i \geq 0\}$

H-polyhedron:

$P(A,z) = \{x \in \mathbb{R}^d : Ax \leq z\}$

V-polyhedron:

$P = \mathrm{conv}(V) + \mathrm{cone}(Y)$

Main Theorem for Polyhedra

H-polyhedra = V-polyhedra

Sketch of Proof:

$\supseteq$: Let $P = \mathrm{conv}(V) + \mathrm{cone}(Y)$ be a V-polyhedron

$= \{x \in \mathbb{R}^d \mid \exists \lambda \in \mathbb{R}^n, \mu \in \mathbb{R}^m : x = V\lambda + Y\mu,\ \sum \lambda_i = 1,\ \lambda_i \geq 0,\ \mu_i \geq 0\}$

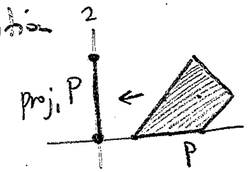
Let $Q = \left\{ \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} \in \mathbb{R}^{d+n+m} \,\middle|\, x = V\lambda + Y\mu : \sum \lambda_i = 1, \lambda_i \geq 0, \mu_i \geq 0 \right\}$

Note:
- Q is an H-polyhedron
- P = projection of Q

So enough to show

proj(H-polyhedron) = H-polyhedron.

We do this by Fourier-Motzkin elimination



$P: \begin{bmatrix} -x_1 + x_2 \leq -1 \\ x_1 + x_2 \leq 5 \\ x_1 - x_2 \leq 3 \\ -x_2 \leq 0 \end{bmatrix} \longrightarrow \begin{array}{c} -x_2 \leq 0 \\ (x_2+1) \leq x_1 \leq \begin{pmatrix} -x_2+5 \\ x_2+3 \end{pmatrix} \end{array}$

$\mathrm{Proj}_1 P: \begin{bmatrix} -x_2 \leq 0 \\ x_2 + 1 \leq -x_2 + 5 \\ x_2 + 1 \leq x_2 + 3 \end{bmatrix} \longrightarrow \boxed{0 \leq x_2 \leq 2}$

$\subseteq$: Let $P = P(A, z)$ be an H-polyhedron

$= \{x \in \mathbb{R}^d : Ax \leq z\}$

Let $Q = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+m} : Ax \leq z \right\}$

affine subspace

Note:
- Q is a V-polyhedron (will show)
- $P = Q \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{d+m} : y = z \right\}$

So enough to show

(V-polyhedron) $\cap$ (affine subspace) = V-polyhedron