Any $n$ polytopes $P_1, \dots, P_n$ have a mixed volume, and we can invert the above eqn ("inclusion-exclusion") to see:

$$\mathrm{Vol}(P_1, \dots, P_n) = \sum_{J \subset [n]} (-1)^{n-|J|} \mathrm{Vol}\Big(\sum_{j \in J} P_j\Big)$$

## An application

**Bezout's Thm:**
If the system of polynomial eqs
$f(x,y)=0 \qquad g(x,y)=0$
has a finite # of sols, then it has $\leq (\deg f)(\deg g)$ sols.
Any "generic" such system has $= (\deg f)(\deg g)$ sols.

Ex: $\deg f = \deg g = 2$

$\leq 4$

Ex: $\begin{cases} y = P(x) \\ y = 0 \end{cases}$ $\leq \deg P$

But e.g., $\begin{cases} a_1 + a_2 x + a_3 xy + a_4 y = 0 \\ b_1 + b_2 x^2 y + b_3 x y^2 = 0 \end{cases}$ never has 6 sols. Why?

**Def** The Newton polygon of $f(x,y) = \sum a_{ij} x^i y^j$ is

$$\mathrm{New}(f) = \mathrm{conv}\{(i,j) \in \mathbb{Z}^2 : a_{ij} \neq 0\}$$

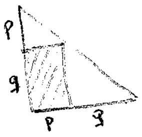
**Bernstein's Thm.**
If the system $f(x,y) = g(x,y) = 0$ has a finite # of sols in $(\mathbb{C}\setminus 0)^2$ then it has $\leq 2\,\mathrm{Vol}(\mathrm{New}(f), \mathrm{New}(g))$ sols.
Any "generic" such system has that many.

Ex above:

$2\mathrm{Vol}(\square, \triangle)$

=  $= 4$

General:

$2\mathrm{Vol}(\triangle_p, \triangle_q) =$ 

$= pq$





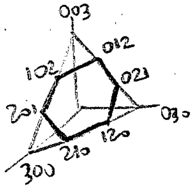
In general,

**Thm**
The system $f_1(x_1, \dots, x_n) = 0, \dots, f_n(x_1, \dots, x_n) = 0$ has
$$n!\,\mathrm{Vol}(\mathrm{New}(f_1), \dots, \mathrm{New}(f_n))$$ isolated
solutions in $(\mathbb{C}\setminus 0)^n$ if coeffs are generic.

**Application:** volume of the permutohedron

Let $\Pi_n = \mathrm{conv}(\text{permuts of } 0, 1, \dots, n-1)$

$$= \Big\{ x \in \mathbb{R}^n : \sum_{k=1}^n x_k = \binom{n}{2}, \quad \sum_{k \in S} x_k \geq \binom{|S|}{2} \quad S \subset [n] \Big\}$$




Let $\Delta_{ij} = \mathrm{conv}(e_i, e_j)$

**Lemma** $\Pi_n = \sum_{1 \leq i < j \leq n} \Delta_{ij}$

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Pf $\Delta_{ij} = \{x \in \mathbb{R}^n : x_i + x_j = 1, x_i \geq 0, x_j \geq 0, \text{other } x_k = 0\}$

$$= \left\{ x \in \mathbb{R}^n : \begin{array}{l} \sum_{k=1}^n x_k = 1 \\ \sum_{k \in S} x_k \geq \begin{cases} 1 & \text{if } i,j \in S \\ 0 & \text{otherwise} \end{cases} \end{array} \right\}$$

Remember: if $c \cdot x$ takes min val $p$ in $P$, $q$ in $Q$

$\Rightarrow$ it takes min val $p+q$ in $P+Q$

So in $Q = \sum_{1 \leq i < j \leq n} \Delta_{ij}$, $\sum_{k=1}^n x_k = \#\text{ of pairs } i,j = \binom{n}{2}$

$\sum_{k \in S} x_k \geq \#\text{ of pairs } i,j \text{ in } S = \binom{|S|}{2}$

So $Q \subseteq \Pi_n$. Exercise: $Q \supseteq \Pi_n$.