More on Volumes

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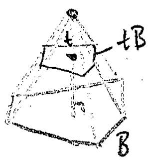
Prop Let P be a d-dim pyramid with base B and height 1. Then $\mathrm{Vol}_d P = \frac{1}{d}(\mathrm{Vol}_{d-1} B)$.

Pf.



$$\mathrm{Vol}_d P = \int_0^h \mathrm{Vol}_{d-1}(tB)\, dt$$
$$= \int_0^h t^{d-1}\,\mathrm{Vol}_{d-1}(B)\, dt = \mathrm{Vol}_{d-1}(B)\left.\frac{t^d}{d}\right]_0^1 \quad \blacksquare$$

Cor Let $P = \mathrm{conv}(0, v_1, \dots, v_d) \subset \mathbb{R}^d$ be a simplex. Then
$$\mathrm{Vol}_d P = \frac{1}{d!}\left|\det\begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix}\right|$$

Pf First let $P_d = \mathrm{conv}(0, e_1, \dots, e_d) \to \det = 1$

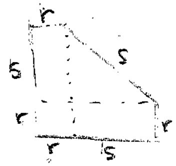
Note $P_d =$  so $\mathrm{Vol}_d P_d = \frac{1}{d}\mathrm{Vol}_{d-1} P_{d-1} = \cdots = \frac{1}{d!}$

Now for general P, use the change of basis $x = Au$, $A = [v_1 \cdots v_d]$, $A(P_d) = P$

$$\int_P 1\, dx = \int_{P_d} 1\,|\det A|\, du = \frac{|\det A|}{d!} \quad \blacksquare$$

Jacobian

Volumes of Minkowski Sums:

Ex. $\mathrm{Vol}(r\square + s\Delta) =$  $= r^2 + \frac{s^2}{2} + 2rs$

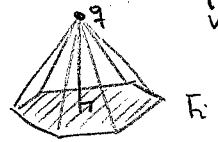
$$= r^2\,\mathrm{Vol}(\square) + s^2\,\mathrm{Vol}(\Delta) + 2rs\,\text{"}\mathrm{Vol}(\square, \Delta)\text{"}$$

Theorem $\mathrm{Vol}_d(rP + sQ)$ is a homogeneous poly in r, s of degree d. We write
$$\mathrm{Vol}_d(rP + sQ) = \sum_{i=0}^{d} \binom{d}{i} \underbrace{\mathrm{Vol}(P^i, Q^{d-i})}_{\text{"mixed volumes"}} r^i s^{d-i}$$

Plan: Induct on d.
Tool:

Prop. Let P be a d-polytope in $\mathbb{R}^d$ with facet description $P = \{x : a_i \cdot x \le b_i \quad i = 1, \dots, m\}$. Let F_i be the facet $a_i \cdot x = b_i$.
Then
$$\mathrm{Vol}_d P = \frac{1}{d}\sum_{i=1}^m b_i\,\mathrm{Vol}_{d-1} F_i, \qquad \sum_{i=1}^m \mathrm{Vol}_{d-1} F_i\, a_i = 0.$$
vectors

Pf Let $q \in \mathrm{int}\, P$. Let $P_i = \mathrm{conv}(F_i, q) =$



$$\mathrm{Vol}_d P = \sum_i \mathrm{Vol}_d P_i$$
$$= \sum_i \frac{1}{d}(\mathrm{Vol}_{d-1} F_i)\cdot h_i = \sum \frac{\mathrm{Vol}_{d-1} F_i}{d}(b_i - a_i \cdot q)$$
$$= \frac{1}{d}\sum_i b_i\,\mathrm{Vol}_{d-1} F_i - \left(\frac{1}{d}\sum \mathrm{Vol}_{d-1} F_i\, a_i\right)\cdot q \quad \blacksquare$$

Lemma. $(rP + sQ)_a = rP_a + sQ_a$

Pf Say $a \cdot x \le p$ in P with eq for $x \in P_a$
$a \cdot y \le q$ in Q with eq for $y \in Q_a$

Then for $rx + sy \in rP + sQ$,
$a \cdot (rx + sy) \le rp + sq$ with eq for $rx + sy \in rP_a + sQ_a$ $\blacksquare$

Pf of Thm.

Let $rP + sQ$ have facets $F_i : a_i x \le b_i$.

If P has $a_i \cdot x \le p_i$
Q has $a_i \cdot x \le q_i$ $\rightarrow rP + sQ$ has $a_i \cdot x \le \underbrace{rp_i + sq_i}_{b_i}$

Then
$$\mathrm{Vol}_d(rP + sQ) = \frac{1}{d}\sum_{i=1}^m b_i\,\mathrm{Vol}_{d-1} F_i$$

$b_i = rp_i + sq_i$ linear in r, s; $\mathrm{Vol}_{d-1} F_i = \mathrm{Vol}_{d-1}(rP_{a_i} + sQ_{a_i})$ homog of deg $d-1$ in r, s. $\blacksquare$

Similarly,
$$V_d(\lambda_1 P_1 + \cdots + \lambda_m P_m) = \sum_{i_1 + \cdots + i_m = d} \binom{d}{i_1 \cdots i_m} \mathrm{Vol}(P_1^{i_1}, \dots, P_m^{i_m})\, \lambda_1^{i_1} \cdots \lambda_m^{i_m}$$