From lattice points to faces:

Theorem (Euler relation)
If P is a d-polytope with f_k k-faces,
$$f_0 - f_1 + f_2 - f_3 \cdots + (-1)^d f_d = 1$$

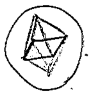
⚠ Not every combin type of polytope is realizable with $\mathbb{Q}$ coordinates! (Perles)

Pf. (For P lattice)
Note $P = \biguplus_{F \text{ face}} F^\circ$, so
$$L_P(t) = \sum_{F \text{ face}} L_{F^\circ}(t) = \sum_{F<P} (-1)^{\dim F} L_F(-t)$$
Recall $[t^0]\, L_P(t) = 1$ (coeff of t^0) so
$$1 = \sum_{F<P} (-1)^{\dim F} (1) = \sum_{k=0}^{d} (-1)^k f_k. \quad \blacksquare$$

- In 3-D, f-vectors of polytopes are classified.
- In dim d, this seems hopeless. The "cd-index" (which encodes flags, not just faces) seems better than the f-vector.
- But for simple/simplicial polytopes, not hopeless!

(A) Linear Relations:

Let the h-vector of P be
$$h_k = f_{k-1} - \binom{d-k+1}{d-k} f_{k-2} + \binom{d-k+2}{d-k} f_{k-3} \cdots + (-1)^k \binom{d}{d-k}$$
Stanley's trick:



```
      1
     1  6   =f
    1  5  12
   1  4  7  8
h= 1  3  3  1
```

Icosahedron

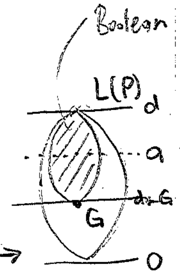
```
      1
     1  12   =f
    1  11  30
   1  10  19  20
h= 1   9   9   1
```


Theorem (Dehn-Sommerville Relations)
If P is a simplicial d-polytope,
$$h_k = h_{d-k} \qquad 0 \le k \le d$$

Fact (Klee)
These are all the linear rels between the f_i's.

Pf This translates to
$$\sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} = \sum_{i=0}^{d-k} (-1)^{d-k-i} \binom{d-i}{k} f_{i-1}$$
which, after some manipulation, is equiv. to (Ex.)
$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1}$$
$$f_{(d-1)-(d-k)} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{(d-1)-(d-i)} \qquad a = d-k,\ b = d-i$$
(*) $$f_a^\Delta = \sum_{b=0}^{a} (-1)^b \binom{d-b}{d-a} f_b^\Delta$$ for the f-vector of P^Δ (simple).

So let's prove (*) for a simple polytope Q.
Note: For each face,
$$L_F(t) = \sum_{G \le F} L_{G^\circ}(t) = \sum_{G \le F} (-1)^{\dim G} L_G(-t)$$
So
$$\sum_{\substack{F \le Q \\ \dim F = a}} L_F(t) = \sum_{\substack{F \le Q \\ \dim F = a}} \sum_{G \le F} (-1)^{\dim G} L_G(-t)$$
$$= \sum_{G \le Q} (-1)^{\dim G} L_G(-t) \Big(\sum_{\substack{F \ge G \\ \dim F = a}} 1 \Big)$$
$$= \sum_{G \le Q} (-1)^{\dim G} L_G(-t) \binom{d - \dim G}{d-a}$$



Now just take coeffs of $[t^0]$ as above $\blacksquare$

(B) Non-Linear Conditions:

The "g-theorem" of Billera-Lee-Stanley completely classifies the f-vectors of simplicial polytopes.