An application:

Lec 25
Oct 22

Pick's Theorem
If a lattice polygon P has I interior lattice points and B boundary lattice points, then
$$\text{Area}(P) = I + B/2 - 1$$

Pf

Let the Ehrhart polynomial be
$$L_P(t) = at^2 + bt + c$$
We saw: $a = \text{Area}(P)$, $c = L_P(0) = 1$.

Also:
$$a+b+c = L_P(1) = I + B$$
$$a-b+c = L_P(-1) = L_{P^\circ}(1) = I$$

$$2a + 2c = 2I + B$$
$$a = \tfrac{1}{2}(2I + B - 2). \quad \blacksquare$$

Another application:
To compute $L_P(t)$ you "just" need d of the numbers:
$$\text{Vol}(P),\ L_P(1),\ L_{P^\circ}(1),\ L_P(2),\ L_{P^\circ}(2), \ldots$$

Another one: "Magic Squares"

Let $H_n(r)$ = # of $n \times n$ $\mathbb{N}$-matrices with row sums and col sums $= r$

Prop $H_n(r)$ is a polynomial in r of degree $(n-1)^2$
$$H_n(-1) = H_n(-2) = \ldots = H_n(-(n-1)) = 0$$
$$(-1)^{n-1} H_n(-m) = H_n(m-n), \quad \text{all } m \geq n$$

Pf Recall the Birkhoff polytope from HW3, Problem 5:
$$B_n = \text{conv}\left\{ \begin{matrix} n\times n \text{ permutation} \\ \text{matrices} \end{matrix} \right\} \subseteq \mathbb{R}^{n^2}$$
$$= \left\{ \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} : \begin{array}{ll} \sum_{j=1}^{n} x_{ij} = 1 & \text{all } i \\ \sum_{i=1}^{n} x_{ij} = 1 & \text{all } j \\ x_{ij} \geq 0 & \text{all } i,j \end{array} \right\}$$

Then clearly
$$H_n(t) = L_{B_n}(t) \quad \text{— a polynomial!}$$
The degree is
$$\dim B_n = n^2 - n - n + 1$$
(row eqs, col eqs, $\Sigma(\text{rows}) = \Sigma(\text{cols})$)

Also
$$H_n(-t) = (-1)^{(n-1)^2} L_{B_n^\circ}(t)$$

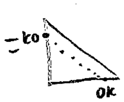
(# of $\mathbb{N}$ $n\times n$ matrices w/ (row sums) = (col sums) = $t-n$) = (# of $\mathbb{Z}_{>0}$ $n\times n$ matrices with (row sums) = (col sums) = t)

$\quad\|$
$H_n(t-n)$ $\quad (=0 \text{ if } t<n)$

Remark If P is a rational polytope (rational vertices) then $L_P(t)$ is a quasipolynomial — a polynomial for $t \equiv i \pmod n$ $(0 \leq i \leq n)$ (some n)

Ex: $P =$ 

$(2k)P =$  $\rightarrow L_P(n) = \dfrac{\frac{n}{2}(\frac{n}{2}+1)}{2}$ $\quad n$ even

$(2k+1)P =$  $\rightarrow L_P(n) = \dfrac{\frac{n-1}{2}(\frac{n-1}{2}+1)}{2}$ $\quad n$ odd

Similar pf, reciprocity, etc.