We saw the Ehrhart polynomial satisfies:

$$L_P(t) = |tP \cap \mathbb{Z}^d| \qquad t = 0,1,2,\dots$$

Recall $P^\circ = \operatorname{relint} P$

Theorem (Ehrhart reciprocity)
P lattice d-polytope

$$L_P(-t) = (-1)^d L_{P^\circ}(t)$$

First for cones:

Theorem (Stanley reciprocity)
K rational d-cone with apex at 0

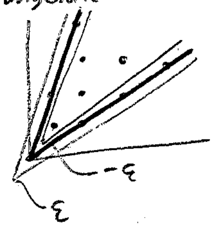
$$\sigma_K\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d}\right) = (-1)^d \sigma_{K^\circ}(z_1,\dots,z_d)$$

(as rational functions in $z_1,\dots,z_d$)

Ex: $K =$  $\sigma_K = \frac{1}{(1-z_1)(1-z_2)}$ $\qquad \sigma_{K^\circ} = \frac{z_1 z_2}{(1-z_1)(1-z_2)}$

$$\frac{1}{(1-1/z_1)(1-1/z_2)} = \frac{z_1 z_2}{(1-z_1)(1-z_2)}$$

Pf Enough for simplicial cones, then "just triangulate".
Last time we saw:
Shift by a tiny irrational vector $\varepsilon \in -K$:



$$\sigma_{\varepsilon+K} = \sigma_K$$

But also notice, by the same argument

$$\sigma_{-\varepsilon+K} = \sigma_{K^\circ}$$

So:

$$\sigma_K\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d}\right) = \sigma_{\varepsilon+K}\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d}\right) = \frac{\sigma_{\varepsilon+\Pi}\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d}\right)}{\prod_{i=1}^{d}\left(1-\frac{1}{z^{v_i}}\right)}$$

$$= (-1)^d \frac{\prod_i z^{v_i}\, \sigma_{\varepsilon+\Pi}\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d}\right)}{\prod(1-z^{v_i})}$$

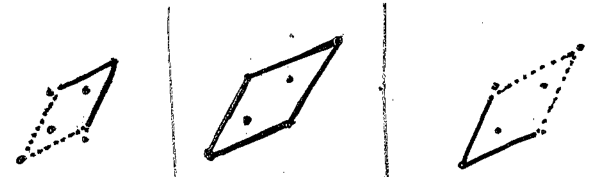
$$\sigma_{K^\circ}(z_1,\dots,z_d) = \frac{\sigma_{\varepsilon+\Pi}(z_1,\dots,z_d)}{\prod(1-z^{v_i})}$$

So we need:

$$\sigma_{-\varepsilon+\Pi}(z_1,\dots,z_d) \stackrel{?}{=} z^{v_1+\dots+v_d}\, \sigma_{\varepsilon+\Pi}\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d}\right)$$

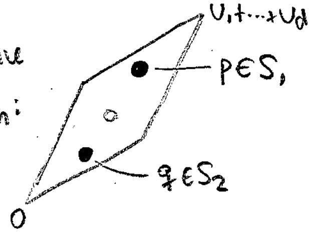
$$\sum_{p \in (-\varepsilon+\Pi)\cap\mathbb{Z}^d} z^p = \sum_{q \in (\varepsilon+\Pi)\cap\mathbb{Z}^d} z^{v_1+\dots+v_d-q}$$

and this is:



$S_1 = (-\varepsilon+\Pi)\cap\mathbb{Z}^d \qquad \Pi\cap\mathbb{Z}^d \qquad (\varepsilon+\Pi)\cap\mathbb{Z}^d = S_2$

So I have a bijection:



$$p = v_1+\dots+v_d - q$$

∎

Pf of Ehrhart reciprocity:

P d-polytope
$K = \operatorname{cone}(P)$ $(d+1)$-cone.

$$\sigma_K\left(\tfrac{1}{z_1},\dots,\tfrac{1}{z_d},\tfrac{1}{z_{d+1}}\right) = (-1)^{d+1}\sigma_{K^\circ}(z_1,\dots,z_d,z_{d+1})$$

$$\sigma_K(1,\dots,1,1/z) = (-1)^{d+1}\sigma_{K^\circ}(1,\dots,1,z)$$

$$\mathrm{Ehr}_P(1/z) = (-1)^{d+1}\mathrm{Ehr}_{P^\circ}(z)$$

Lemma

$$-\sum_{t<0} L_P(t) z^{-t} = \sum_{t\geq 0} L_P(t) z^{-t}$$

as rational functions. (True for any polynomial $f(t)$.)

$$= (-1)^{d+1}\sum_{t\geq 0} L_{P^\circ}(t) z^t$$

$\Rightarrow L_P(-t) = (-1)^d L_{P^\circ}(t)$. ∎

Pf Show it for $f(t) = \binom{t+d}{d}$ $\quad d = 0,1,\dots$
which are a basis for the space of polynomials. ∎