Thm (Stanley's Nonnegativity Theorem)

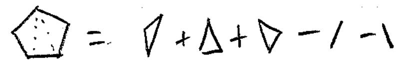
Let P be a lattice polytope with

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + \cdots + h_0}{(1-z)^{d+1}}$$

Then $h_0, h_1, \ldots, h_d \geq 0$

Lec 23
Oct 13

Pf We proved this for simplices. Can we "just translate"?



The "−" signs give us trouble!

Trick: Translate $\mathrm{cone}(P) = C_1 \cup \cdots \cup C_k$

Shift by a tiny "irrational vector" $\varepsilon \in -\mathrm{Cone}(P)$, $\varepsilon \notin \partial C_i$ (e.g. $\varepsilon = v + \frac{1}{N}(1, \pi, \pi^2, \ldots, \pi^{d+1})$)

$$\varepsilon + \mathrm{cone}(P) = (\varepsilon + C_1) \cup \ldots \cup (\varepsilon + C_k)$$

so that,

$$(\varepsilon + \mathrm{Cone}(P)) \cap \mathbb{Z}^{d+1} = \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}$$

$Ax \leq A\varepsilon$ ($A\varepsilon > 0$ tiny) $\qquad$ $Ax \leq 0$

The point is that the boundary of $\varepsilon + C_i$ cannot contain lattice pts:

if $p \in \partial(\varepsilon + C_i) = \varepsilon + \partial C_i$

then $p - \varepsilon$ satisfies some integer equation

$$a(p - \varepsilon) = 0 \qquad \underbrace{ap}_{\mathbb{Z}} = \underbrace{a\varepsilon}_{\text{irrational, nonzero.}}$$

So

$$\mathrm{Cone}(P) \cap \mathbb{Z}^{d+1} = (\varepsilon + \mathrm{cone}(P)) \cap \mathbb{Z}^{d+1} = \biguplus_i (\varepsilon + C_i) \cap \mathbb{Z}^{d+1}$$

So

$$\sigma_{\mathrm{Cone}(P)}(1, \ldots, 1, z) = \sum_i \sigma_{\varepsilon + C_i}(1, \ldots, 1, z) = \sum_i \frac{\text{gf for pts in } \Pi_i}{(1-z)^{d+1}} \quad \blacksquare$$

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$(h_0, \ldots, h_d)$ is the "h^*-vector" of P. Many open problems about it. What do they look like?
Mustata-Payne: topological interpretation

Lemma The Ehrhart poly. of P is

$$L_P(t) = h_0\binom{t+d}{d} + h_1\binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d\binom{t}{d}$$

Pf

$$\sum_{t \geq 0} L_P(t) z^t = \frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^{d+1}} = \left(\sum_{i=0}^{d} h_i z^i\right)\left(\sum_{j \geq 0} \binom{j+d}{d} z^j\right)$$

From $\mathrm{Ehr}_{\Delta_{d+1}}$

So coeff of z^t is $\displaystyle\sum_{i+j=t} h_i \binom{j+d}{d}$. $\blacksquare$

Some easy facts:

$h_0 = 1$ $\qquad$ $h_1 = |P \cap \mathbb{Z}^d| - d - 1$ $\qquad$ others??

Now write

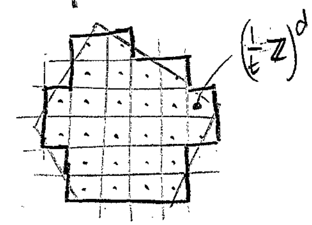
$$\boxed{L_P(t) = C_d t^d + \cdots + C_1 t + C_0}$$

Also easy:

$C_0 = 1$ $\qquad$ others?

Prop $C_d = \mathrm{vol}\, P$ $\qquad$ $\left(\mathrm{vol}\, P = \int_P dx\right)$

"Pf". We can approximate $\mathrm{vol}\, P$ by little boxes of sidelength $1/t$ around the points of P in the lattice $\left(\frac{1}{t}\mathbb{Z}\right)^d$



$$\mathrm{vol}\, P = \lim_{t\to\infty} \left|P \cap \left(\tfrac{1}{t}\mathbb{Z}\right)^d\right| \cdot \left(\tfrac{1}{t}\right)^d$$

$$= \lim_{t\to\infty} \frac{|tP \cap \mathbb{Z}^d|}{t^d} = \lim_{t\to\infty} \frac{L_P(t)}{t^d} = C_d \quad \blacksquare$$

Also $C_{d-1} = \frac{1}{2}$ (normalized surface area (P)) $\qquad$ others?

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