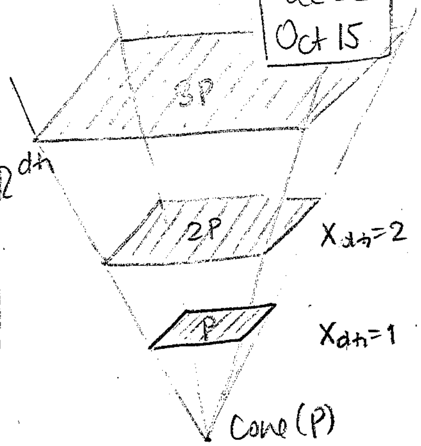
From cones to polytopes.

Lec 22
Oct 15

$$P = \mathrm{conv}\{v_1, \dots, \overset{\nearrow \mathbb{Z}^d}{v_n}\} \subset \mathbb{R}^d$$

$$\Rightarrow \mathrm{cone}(P) = \mathrm{cone}\left\{ \underset{\overset{\shortparallel}{w_1}}{\begin{bmatrix} v_1 \\ 1 \end{bmatrix}}, \dots, \underset{\overset{\shortparallel}{w_n}}{\begin{bmatrix} v_n \\ 1 \end{bmatrix}} \right\} \subset \mathbb{R}^{d+1}$$



Recall:

$$\boxed{\begin{aligned} L_P(t) &= |tP \cap \mathbb{Z}^d| = \sigma_{tP}(1) \\ \mathrm{Ehr}_P(z) &= \sum_{t \geq 0} L_P(t) z^t \end{aligned}}$$

Check: $\mathrm{Cone}(P) \cap \begin{pmatrix} \text{hyperplane} \\ x_{d+1} = t \end{pmatrix} \cong tP$

So

$$\sigma_{\mathrm{Cone}(P)}(z_1, \dots, z_{d+1}) = \sum_{m \in \mathrm{Cone}(P)} z^m$$

$$= \sum_{\substack{m \in \mathrm{Cone}(P) \\ m_{d+1} = 0}} z^m + \sum_{\substack{m \in \mathrm{Cone}(P) \\ m_{d+1} = 1}} z^m + \sum_{\substack{m \in \mathrm{Cone}(P) \\ m_{d+1} = 2}} z^m + \dots$$

$$= 1 + \sigma_P(z) z_{d+1} + \sigma_{2P}(z) z_{d+1}^2 + \dots$$

So

$$\boxed{\begin{aligned} \sigma_{\mathrm{Cone}(P)}(1, 1, \dots, 1, z_{d+1}) &= 1 + L_P(1) z_{d+1} + L_P(2) z_{d+1}^2 + \dots \\ &= \mathrm{Ehr}_P(z_{d+1}) \end{aligned}}$$

Now assume P is a simplex.

$$\sigma_{\mathrm{Cone}(P)}(1, 1, \dots, 1, z_{d+1}) = \frac{\sigma_\Pi(1, \dots, 1, z_{d+1})}{\left(1 - (1, \dots, 1, z)^{w_1}\right) \dots \left(1 - (1, \dots, 1, z)^{w_{d+1}}\right)}$$

$$= \sigma_\Pi(1, \dots, 1, z_{d+1}) / (1 - z)^{d+1}$$

So

$$\mathrm{Ehr}_P(z_{d+1}) = \frac{\sum_{k \geq 0} h_k z_{d+1}^k}{(1 - z)^{d+1}}$$

where

$h_k = \#$ of lattice pts in Π with $x_{d+1} = k$.

But

$$x \in \Pi \Rightarrow x = \sum_{i=1}^{d+1} \lambda_i w_i \qquad 0 \leq \lambda_i < 1$$

$$\begin{bmatrix} x \\ x_{d+1} \end{bmatrix} = \sum_{i=1}^{d+1} \lambda_i \begin{bmatrix} v_i \\ 1 \end{bmatrix}$$

$$x_{d+1} = \sum_{i=1}^{d+1} \lambda_i < d+1$$

So

$$\boxed{\mathrm{Ehr}_P(z) = \frac{h_0 + h_1 z + \dots + h_d z^d}{(1 - z)^{d+1}}}$$

Lemma Let $f: \mathbb{N} \to \mathbb{C}$ be a function.

$\begin{pmatrix} f(n) \text{ is a polynomial} \\ \text{of degree } d \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sum_{n \geq 0} f(n) z^n = \frac{g(z)}{(1-z)^{d+1}} \\ \text{where } g \text{ is a polynomial} \\ \text{of degree } d \text{ with } g(1) \neq 0 \end{pmatrix}$

Pf: HW4.

Corollary: $L_P(n)$ is a polynomial in n of degree d, for P a simplex.

$\boxed{\text{Corollary: } L_P(n) \text{ is a polynomial in } n \text{ of degree } d \text{ for any } d\text{-polytope } P \text{ with integer coefficients}}$ "Ehrhart polynomial"

Pf: "Just triangulate". ∎

Also: $\boxed{\text{The } h_i \text{ above are } \geq 0}$ "h^*-vector" of P