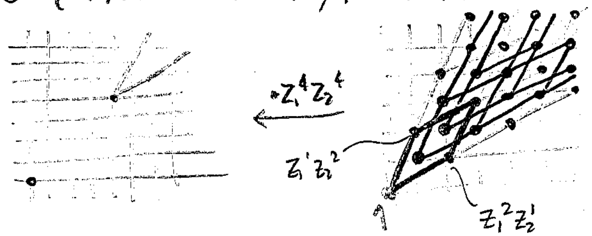
Generating fns for cones:

For $S \subset \mathbb{R}^d$ let

$$\sigma_S(z) = \sum_{m \in S \cap \mathbb{Z}^d} z^m \qquad \text{So} \qquad \sigma_S(1) = |S \cap \mathbb{Z}^d|$$

Easier for cones than for polytopes...

Ex: $C = \{(v,f) : v \le 2f-4,\ f \le 2v-4\}$



$$\frac{1}{(1-z_1z_2^2)(1-z_1^2z_2)} + \frac{z_1z_2}{\text{same}} + \frac{z_1^2z_2^2}{\text{same}}$$

$$\sigma_C(z) = \frac{z_1^4z_2^4\,(1 + z_1z_2 + z_1^2z_2^2)}{(1-z_1z_2^2)(1-z_1^2z_2)}$$

Theorem Let $K = \text{cone}\{w_1, \dots, w_d\} \subset \mathbb{R}^d$
$= \{\lambda_1 w_1 + \dots + \lambda_d w_d : \lambda_1, \dots, \lambda_d \ge 0\}$
be a simplicial cone, where $w_1, \dots, w_d \in \mathbb{Z}^d$.
Let $\Pi = \{\mu_1 w_1 + \dots + \mu_d w_d : 0 \le \mu_1, \dots, \mu_d < 1\}$ be the fundamental parallelepiped of K. Then, for $v \in \mathbb{Z}^d$,

$$\sigma_{v+K} = \frac{z^v\, \sigma_\Pi(z)}{(1-z^{w_1})\cdots(1-z^{w_d})}$$

Pf A term in LHS is z^m for $m \in v+K$

Say $m = v + \lambda_1 w_1 + \dots + \lambda_d w_d \qquad \lambda_i \ge 0$

Write $\lambda_i = \underset{\text{integer}}{a_i} + \underset{\text{in } [0,1)}{\mu_i}$

$\Rightarrow m = v + \underbrace{(\mu_1 w_1 + \dots + \mu_d w_d)}_{p \in P} + \underbrace{(a_1 w_1 + \dots + a_d w_d)}_{\text{any } a_i \ge 0}$

and from p, a_i, we can recover m. So

$$\sigma_{v+K} = \sum_{m \in v+K} z^m = \sum_{p \in P} \sum_{a_i \ge 0} z^{v+p+\sum a_i w_i}$$

$$= z^v \Big(\sum_{p \in P} z^p\Big)\Big(\sum_{a_1 \ge 0} z^{a_1 w_1}\Big) \cdots \Big(\sum_{a_d \ge 0} z^{a_d w_d}\Big)$$

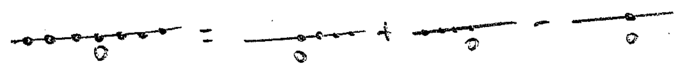
$$= z^v\, \sigma_\Pi(z)\, \frac{1}{1-z^{w_1}} \cdots \frac{1}{1-z^{w_d}} \qquad \blacksquare$$

Say a cone K is *pointed* if $K = \overset{\text{(apex)}}{v} + \text{cone}\{w_i\}$ where the w_i are on the positive side of some hyperplane H

Corollary For any pointed cone K, the generating function σ_K is rational. Analysis: convergent in some domain, dep. on H.

Pf Triangulate into simplicial cones, use previous theorem

Why pointed? What about $K = \text{cone}\{-1, 1\} = \mathbb{R}$?



$$\sigma_{\mathbb{R}}(z) = \frac{1}{1-z} + \frac{1}{1-1/z} - 1 = \frac{1}{1-z} + \frac{z}{z-1} - 1 = 0$$

In general, $\sigma_K(z) = 0$ for any K not pointed.

⚠ Is this nonsense? One series converges for $|z| < 1$
Other for $|\frac{1}{z}| < 1$
So the domains of convergence are disjoint!

It is not nonsense, but needs to be made precise, via "algebra of rational polyhedra" $P(\mathbb{Q}^d) = \text{span}\{[P] : P \subset \mathbb{R}^d$ rational? polyhedra$\}$

$\Rightarrow$ There is $P(\mathbb{Q}^d) \xrightarrow{\sigma} \mathbb{C}(z_1, \dots, z_d)$