A very important example: Partition Functions

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Ex 1:

How many ways to pay n cents with US coins? (1, 5, 10, 25)

$$f(n) = \#\{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^d : \begin{array}{c} m_1, m_2, m_3, m_4 \geq 0 \\ m_1 + 5m_2 + 10m_3 + 25m_4 = n \end{array}\}$$

$$= |nP \cap \mathbb{Z}^d| \qquad P = \{m \in \mathbb{Z}^4 : m \geq 0 : m_1 + 5m_2 + \dots = 1\}$$

Note:

$$\left(\frac{1}{1-z}\right)\left(\frac{1}{1-z^5}\right)\left(\frac{1}{1-z^{10}}\right)\left(\frac{1}{1-z^{25}}\right) = \left(\sum_{m_1 \geq 0} z^{m_1}\right)\left(\sum_{m_2 \geq 0} z^{5m_2}\right)\left(\sum_{m_3 \geq 0} z^{10m_3}\right)\left(\sum_{m_4 \geq 0} z^{25m_4}\right)$$

$$= \sum_{m \geq 0} z^{m_1 + 5m_2 + 10m_3 + 25m_4}$$

$$= \sum_{n \geq 0} f(n) z^n$$

So

$$\boxed{f(n) = [z^n] \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}}$$

$$= [z^n]\left(\frac{A}{1-z} + \frac{B_1}{1-\omega_5 z} + \dots + \frac{B_5}{1-\omega_5^4 z} + \frac{C_1}{1-\omega_{10} z} + \dots + \frac{D_1}{1-\omega_{25} z} + \dots\right)$$

and we can use this to "compute" $f(n)$.

More generally:

Let $A = \{a_1, \dots, a_n\} \subset \mathbb{Z}^d$

The partition function

$$\phi_A(b) = \#\text{ of ways to "partition" } b \text{ into } a_i\text{'s}$$

$$= \#\{(x_1, \dots, x_n) \in \mathbb{Z}^d : x \geq 0,\ a_1x_1 + \dots + a_nx_n = b\}$$

$$= |P \cap \mathbb{Z}^d|$$

where $P = \{x : x \geq 0, Ax = b\}$

Notation: $b \in \mathbb{Z}^d \rightsquigarrow z^b = z_1^{b_1} \cdots z_d^{b_d}$

Then

Theorem

$$\boxed{\phi_A(b) = [z^b] \frac{1}{(1-z^{a_1}) \cdots (1-z^{a_n})}}$$

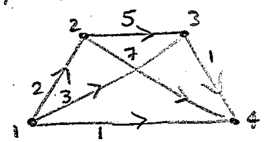
In theory, simple

In practice, hard to compute. lots of interesting theory (alg, comb, anal), and interesting open problems.

Ex 2 Rep. th. of sl_n → Kostant's partition function

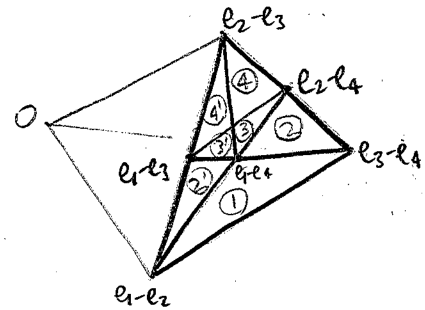
$$A_{n-1} = \{e_i - e_j : 1 \leq i < j \leq n\} \qquad \text{"root system"}$$

$\phi_{A_{n-1}}(b) = \#$ of "(-b)-leaky flows" of K_n:



$-b = (-6, -10, +7, +9)$

(leaks at each vertex)



$$\phi_{A_3}(b) = \begin{cases} (b_1+b_2+3)(b_1+b_2+2)(b_1+b_2+1)/6 & \text{in } 1 \\ (b_1+1)(b_1+2)(b_1+3b_2+3)/6 & \text{in } 2 \\ 1 + \frac{11}{6}b_1 + \dots - \frac{1}{2}b_3^3 \quad (12 \text{ terms}) & \text{in } 3 \\ (b_1+2)(b_1+1)(2b_1+3b_2+3b_3+3) & \text{in } 4 \end{cases}$$

- General facts:
  - piecewise (quasi) polynomial
  - chambers determined by simplices spanned by A
  - "wall-crossing" formulas
- Not entirely explained:
  - nice formulas/factorization
  - "chamber merging"