Ok, so what is a polytope?

lecture 2
aug 27/10

Def. A <u>polytope</u> is

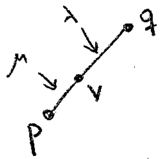
convex hull

$$P = \operatorname{conv}\{v_1, \dots, v_n\}$$
$$= \{\lambda_1 v_1 + \dots + \lambda_n v_n : \Sigma \lambda_i = 1,\ \lambda_i \geq 0\}$$

for $v_1, \dots, v_n \in \mathbb{R}^d$

How does this match intuition?

1-D

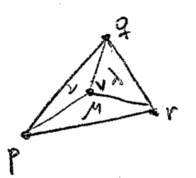


$$\vec{pv} = \mu \vec{pq}$$
$$(v-p) = \mu(q-p)$$

$$\boxed{v = \lambda p + \mu q \quad \lambda + \mu = 1 \quad \lambda, \mu \geq 0}$$

$$\lambda = \frac{|vq|}{|pq|} \qquad \mu = \frac{|pv|}{|pq|}$$

2-D

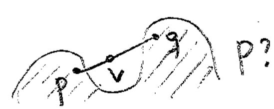


$$\lambda = \frac{\text{area}(vqr)}{\text{area}(pqr)}, \ \mu = \dots, \ \nu = \dots$$

$$\boxed{v = \lambda p + \mu q + \nu r \quad \lambda + \mu + \nu = 1 \quad \lambda, \mu, \nu \geq 0}$$

(exercise)

Prop. P is convex



Pf. Let $p, q \in P$, $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$, $\lambda, \mu \geq 0$. Let $v = \lambda p + \mu q$

Say $p = \lambda_1 v_1 + \dots + \lambda_n v_n$, $q = \mu_1 v_1 + \dots + \mu_n v_n$ $\quad \Sigma \lambda_i = 1\ \lambda_i \geq 0$, $\Sigma \mu_i = 1\ \mu_i \geq 0$

Then $v = \lambda p + \mu q$
$$= (\lambda\lambda_1 + \mu\mu_1) v_1 + \dots + \underbrace{(\lambda \lambda_n + \mu \mu_n)}_{\geq 0} v_n$$

where $\Sigma(\lambda\lambda_i + \mu\mu_i) = \lambda \Sigma \lambda_i + \mu \Sigma \mu_i = \lambda + \mu = 1$

so $v \in P$. ∎

Observation: Convex sets are closed under arbitrary convex combinations:

P convex $\Rightarrow$ If $p_1, \dots, p_n \in P$ $\quad \lambda_1, \dots, \lambda_n \in \mathbb{R}$ $\quad \Sigma \lambda_i = 1$, $\lambda_i \geq 0$

then $\lambda_1 p_1 + \dots + \lambda_n p_n \in P$.

(Exercise)

From this it easily follows that

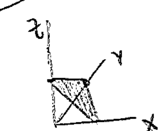
Prop. $P = \operatorname{conv}(v_1, \dots, v_n)$ is the smallest convex set containing $v_1, \dots, v_n$.
Any other convex Q containing $v_1, \dots, v_n$ must in fact contain P.

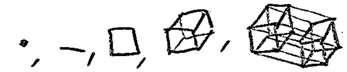
<u>Examples of polytopes</u>

Standard d-simplex: 

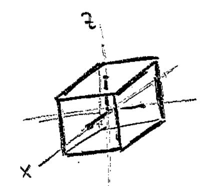
unit vectors

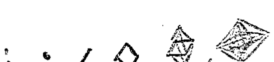
$$\Delta_d = \operatorname{conv}(e_1, \dots, e_{d+1}) \subset \mathbb{R}^{d+1}$$
$$= \{x \in \mathbb{R}^{d+1} : \Sigma x_i = 1,\ x_i \geq 0\}$$



d-cube: 

$$C_d = \operatorname{conv}\{1, -1\}^d$$
$$= \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1\}$$



d-crosspolytope: 

$$\diamond_d = \operatorname{conv}\{e_1, -e_1, \dots, e_d, -e_d\}$$
$$= \{x \in \mathbb{R}^d : a x \leq 1 \text{ for all } a \in \{1,-1\}^d\}$$