Counting Lattice Points in Polytopes

("Measuring polytopes discretely")

Goal:
Given $P \subset \mathbb{R}^d$, compute $|P \cap \mathbb{Z}^d|$.

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Let

- $L_P(t) = |tP \cap \mathbb{Z}^d|$ for $t \in \mathbb{N}$.
- $L_{P^o}(t) = |tP^o \cap \mathbb{Z}^d|$ (P^o = relint P)
- $\mathrm{Ehr}_P(z) = \sum_{t \geq 0} L_P(t) z^t$

$\boxed{① \ P = \square_d = [0,1]^d}$

$tP = \{x \in \mathbb{R}^d : 0 \leq x_i \leq t, \text{ all } i\}$

$tP^o = \{x \in \mathbb{R}^d : 0 < x_i < t, \text{ all } i\}$

so $\boxed{L_{\square_d}(t) = (t+1)^d}$ $\boxed{L_{\square_d^o}(t) = (t-1)^d}$

$$\mathrm{Ehr}_{\square_d}(z) = \sum_{t \geq 0} (t+1)^d z^t$$

$d=0$: $\mathrm{Ehr}_{\square_0}(z) = \sum_{t\geq 0} (t+1)^0 z^t = \sum_{t \geq 0} z^t = \frac{1}{1-z}$

$d=1$: $\mathrm{Ehr}_{\square_1}(z) = \sum_{t \geq 0} (t+1)^1 z^t = \frac{d}{dz}\left(\sum_{t \geq 0} z^{t+1}\right) = \frac{1}{(1-z)^2}$

$d=2$: $\mathrm{Ehr}_{\square_2}(z) = \sum_{t \geq 0} (t+1)^2 z^t = \frac{d}{dz}\left[\sum_{t\geq 0} (t+1) z^{t+1}\right]$

$= \frac{d}{dz}\left[\frac{z}{(1-z)^2}\right] = \frac{1+z}{(1-z)^3}$

$d=3$: $\frac{1+4z+z^2}{(1-z)^4}$ $d=4$: $\frac{1+11z+11z^2+z^3}{(1-z)^5}$

Prop $\boxed{\mathrm{Ehr}_{\square_d}(z) = \frac{A(d,1)x^0 + \cdots + A(d,d)x^{d-1}}{(1-x)^{d+1}}}$

where $A(d,k)$ = # of permutations π of $[d]$ with $k-1$ "descents"

i such that $\pi(i) > \pi(i+1)$

Observations:

- $L_P(t)$ is a polynomial
- $L_P(-t) = (-1)^d L_{P^o}(t)$
- $\mathrm{Ehr}_P(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}$ ← "$h_P^*(z) = h^*$-polynomial"

$\boxed{② \ P = \Delta_d}$

$tP = \{x \in \mathbb{R}^{d+1} : \sum x_i = t, \ x_i \geq 0\}$

$tP^o = \{x \in \mathbb{R}^{d+1} : \sum x_i = t, \ x_i > 0\}$

So $\boxed{L_{\Delta_d^o}(t)}$ = # of $\mathbb{Z}_{>0}$ sols to $x_1 + \cdots + x_{d+1} = t$

= # of ways to insert d bars: $\overbrace{| \, | \, | \, | \, | \, | \, | \, |}^{t}$

$2 + 3 + 2 = 7$

$\boxed{= \binom{t-1}{d}}$

and $\boxed{L_{\Delta_d}(t)}$ = # of $\mathbb{Z}_{\geq 0}$ sols to $x_1 + \cdots + x_{d+1} = t$

= (# of $\mathbb{Z}_{>0}$ sols to $y_1 + \cdots + y_{d+1} = t + (d+1)$) $\boxed{= \binom{t+d}{d}}$

Note: $\binom{-a}{b} = \frac{(-a)(-a-1)\cdots(-a-(b-1))}{b!} = (-1)^b \frac{(a+b-1)\cdots a}{b!} = (-1)^b \binom{a+b-1}{b}$

So $\boxed{\mathrm{Ehr}_{\Delta_d}(z)} = \sum_{t \geq 0} \binom{t+d}{d} z^t = \sum_{t \geq 0} \binom{-(d+1)}{t} (-1)^t z^t = \boxed{\frac{1}{(1-z)^{d+1}}}$

Also: $L_{\Delta_d}(-t) = \binom{-t+d}{d} = (-1)^d \binom{t-1}{d} = (-1)^d L_{\Delta_d^o}(t)$

Same phenomena!