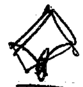
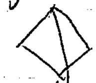
(My) Polytopes Lecture

Weds, 6 October 20[illegible]

Triangulations of d-cross polytope:

$$C_d^\Delta = \operatorname{conv}\{e_1, -e_1, \ldots, e_d, -e_d\} \subset \mathbb{R}^d.$$

eg: $C_2^\Delta =$ , triangs  and .

- Any simplex T we use has $d+1$ vertices, so must contain e_i and $-e_i$ for some i.
- If T ~~contains~~ uses $e_i, -e_i, e_j, -e_j$ for $i \neq j$, have four vertices in the x_i-x_j-plane in $\mathbb{R}^d$. Not possible: ~~these~~ all vertices in a simplex are affinely indep.
- So $T = \operatorname{conv}\{\pm e_1, \ldots, e_i, -e_i, \ldots, \pm e_d\}$ for some i and choice of signs. Say T is of type i.

  eg: $d=5, i=3$, could have $T = \operatorname{conv}\{e_1, e_2, e_3, -e_3, -e_4, e_5\}$.
- $d \cdot 2^{d-1}$ choices for T.

Lemma: All simplices in a triangulation of C_d^Δ have the same type.

PF: If T_1 is of type i, it has $\{e_i, e_{-i}\}$ as an edge.

If T_2 " " " $j \neq i$ " $\{e_j, e_{-j}\}$ as an edge.

These edges meet at $\{0\}$, not a face of either. $\square$

Lemma 2: To cover C_d^{Δ} with simplices of type i, we need all of them, and conversely they do cover it.

PF: Choose arbitrary $(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots \varepsilon_d) \in \{\pm 1\}^{d-1}$.

The point $(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \ldots \varepsilon_d)$ is only in the type i simplex with the same sign pattern.

Conversely, given any point $\underline{x} = (x_1, \ldots x_d) \in C_d^{\Delta}$, choose signs

$$\varepsilon_1 = \mathrm{sign}(x_1), \ldots \varepsilon_{i-1} = \mathrm{sign}(x_{i-1}), \varepsilon_{i+1} = \mathrm{sign}(x_{i+1}), \ldots^{\Delta}$$

to get a simplex of type i containing $\underline{x}$:

$$T_i^{\underline{\varepsilon}} := T_i^{\varepsilon_1 \ldots \varepsilon_{i-1} \varepsilon_{i+1} \ldots \varepsilon_d}. \quad \square$$

d candidates for triangulations of C_d^{Δ}.

THM: All of these are triangulations (in fact regular.)

PF: Choose a type i. ~~Know~~ By Lemma 2,

$$\bigcup_{(\varepsilon_1, \ldots \hat{\varepsilon}_i, \ldots \varepsilon_d) \in \{\pm 1\}^{d-1}} T_i^{\varepsilon_1 \ldots \hat{\varepsilon}_i \ldots \varepsilon_d} = C_d^{\Delta}.$$

Δ or choose arbitrarily if the coord is zero.

If $\varepsilon_1, \ldots \hat{\varepsilon_i}, \ldots \varepsilon_d$ and $\delta_1, \ldots \hat{\delta_i}, \ldots \delta_d$ are two sign patterns, then

$$T_i^{\varepsilon_1, \ldots \hat{\varepsilon_i}, \ldots \varepsilon_d} \cap T_i^{\delta_1 \ldots \hat{\delta_i} \ldots \delta_d}$$

$$= \left\{ \underline{x} \in C_d^\Delta : \begin{cases} x_j = 0 & \text{if } \varepsilon_j \neq \delta_j \\ x_j \geq 0 & \text{if } \varepsilon_j = \delta_j = 1 \\ x_j \leq 0 & \text{if } \varepsilon_j = \delta_j = -1 \\ (x_i & \text{arbitrary}) \end{cases} \right\}$$

$$= \{ \underline{x} \in T_i^{\varepsilon} : x_j = 0 \text{ for } \varepsilon_j \neq \delta_j \}.$$

(since other inequalities hold on all of this simplex)

$=$ intersection of some facets of $T_i^{\varepsilon_1 \ldots \hat{\varepsilon_i} \ldots \varepsilon_d}$ (check!)

$=$ a face of $T_i^{\underline{\varepsilon}}$ since $T_i^{\underline{\varepsilon}}$ is a simplex.

Same for $T_i^{\underline{\delta}}$. $\square$

Why regular?

One reason: $\exists$ a regular triang. by previous lecture.

By symmetry, all of ~~these~~ these d triangs. are regular

or none are. So all are regular.
[actually could even have proved that they're triangs. this way.]

Or constructively: need $\{e_i, -e_i\}$ to appear in the lower hull.
Choose height$(x_1, \ldots, x_n) = 1 - |x_i|$.

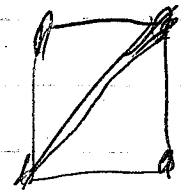
Very non-generic, but check that it works.
[so we have a complete answer, and there are ___very few___ triangs.]

Contrast: $C_d = d$-cube. - lots is unknown:
(reference: Zong "What is known about unit cubes")

THM: $\exists$ a triangulation of C_d with $d!$ simplices.
(the most possible, since min volume of an integral d-simplex is $\frac{1}{d!}$.)

PF sketch: Induct on d.

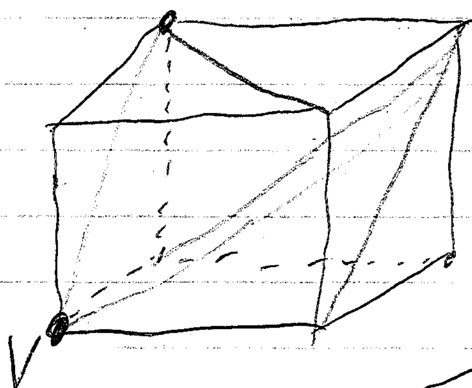
$d=2$:

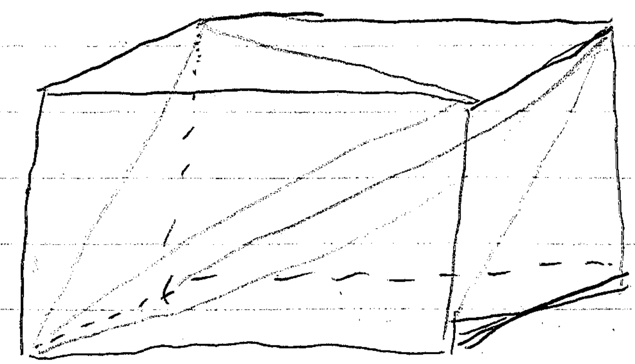


Suppose true for $d-1$.

Choose a vertex v of C_d. Let $F_1, \ldots F_d$ be the facets that don't contain v. For $1 \leq i \leq d$, Let $\{T_{i,j} : 1 \leq j \leq (d-1)!\}$ be a triangulation of F_i. Let $S_{i,j} = \mathrm{conv}(T_{i,j} \cup \{v\})$

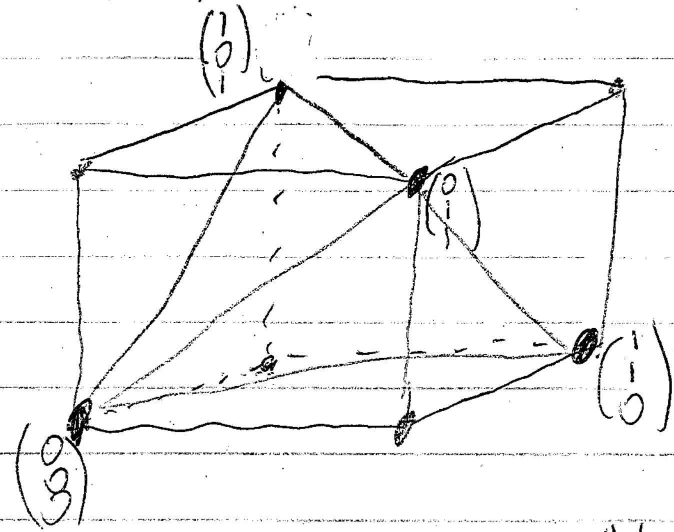
CLAIM: $\{S_{i,j} : 1 \leq i \leq d,\ 1 \leq j \leq (d-1)!\}$ is a triangulation of C_d.





But $\exists$ triangulations with $< d!$ simplices as well.

eg:



$$T = \mathrm{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$$

$$\mathrm{vol}(T) = \frac{1}{6} \cdot \left|\det\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\right| = \frac{1}{3}.$$

Check: $C_3 \setminus T$ is the union of 4 ~~open simpl~~ tetrahedra with disjoint interiors, each of volume $\frac{1}{6}$, and ~~span C_3~~ T and these four form a triangulation.

Open: Asymptotics of $T_d = \min\{\#\text{ simplices in a triangulation of } C_d\}$

Known:

$$\frac{2^d \cdot d!}{(d+1)^{\left(\frac{d+1}{2}\right)}} \underset{\text{all } d}{\le} T_d \underset{\text{suff. large } d}{\le} (0.816^d)\cdot d!$$

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Sequence $\{T_d\}$ begins $1, 2, 5, 16, 67, 308, 1493 \ldots$

even harder: asymptotics of

$N_d :=$ # triangs. of C_d.

See also book *Triangulations: Structures for Algorithms and Applications* by De Loera, Rambau, and Santos.

or paper "Triangulating the κ-cube" by Carl Lee.