Some comments on triangulations

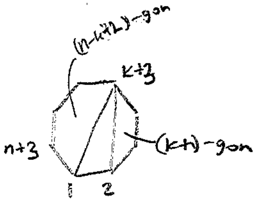
① The n-gon

Let $C_n = \#$ of Δns of the $(n+2)$-gon

$C_0 = 1 \quad C_1 = 1 \quad C_2 = 2 \quad C_3 = 5$

$\boxed{\text{Prop } C_{n+1} = \sum_{k=0}^{n} C_k \, C_{n-k}}$

Pf. If edge 12 is covered by triangle 1,2,k+3, we need to triangulate a (k+2)-gon and a (n-k+2)-gon ∎



$\boxed{\text{Prop } C_n = \frac{1}{n+1}\binom{2n}{n}}$

Pf. We use formal power series:

$$\mathbb{C}[[x]] = \{ \sum_{n\geq 0} a_n x^n : a_n \in \mathbb{C} \}$$

with formal sum, product

Let $C(x) = \sum_{n\geq 0} C_n x^n$.

Then
$$C(x)^2 = \Big(\sum_{n\geq 0} C_n x^n\Big)^2 = \sum_{n\geq 0} \Big(\sum_{k=0}^{n} C_k C_{n-k}\Big) x^n = \sum_{n\geq 0} C_{n+1} x^n$$

$$x\,C(x)^2 = \sum_{n\geq 0} C_{n+1} x^{n+1} = C(x) - 1$$

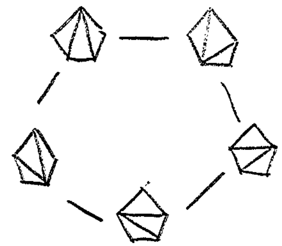
$$x\,C(x)^2 - C(x) + 1 = 0$$

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x} \qquad C(0) = 0$$

$$\Rightarrow C_n = [x^n]\, C(x) = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} = \cdots = \frac{(2n)!}{n!\,(n+1)!}$$ ∎

Say two Δ's are adjacent if they differ by a "flip": 

What is the graph of triangulations:



$\boxed{\text{Thm The graph of } \Delta\text{ns of the } n\text{-gon is the skeleton of a polytope, called the "associahedron"}}$

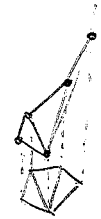
(Stasheff - homotopy theory)

In fact,

- one can define "flips" in any dimension
- the graph of regular Δns of P is the skeleton of the "secondary polytope" of P. (Gelfand-Kapranov-Zelevinsky)

It turns out:

- all Δns of a polygon are regular:
- not all Δns of a general polytope are regular.



② The crosspolytope $\Diamond_n = \text{conv}(\pm e_i : 1\leq i\leq n)$

Simplices need $n+1$ vertices $\to$ must have $e_i, -e_i$
$\to$ can't have $e_i, -e_i, e_j, -e_j$

So a simplex must be like $\text{conv}(e_1, -e_1, \underbrace{\pm e_2, \ldots, \pm e_n}_{2^{n-1} \text{ choices}})$

You: these 2^{n-1} simplices triangulate $\Diamond_n$

$\Rightarrow \Diamond_n$ has n triangulations.

③ The product $\Delta_{n-1} \times \Delta_{m-1}$.

- All Δns have $\binom{m+n}{m}$ simplices
- Not all regular
- Combinatorially described.
- Secondary polytope?

④ Cyclic Polytope $C_d(n)$

nice! still, open problems

⑤ Messier / Open

- cubes
- $\Delta_{n_1} \times \cdots \times \Delta_{n_k}$
- $P \times I$