Complexes, Subdivisions, Triangulations

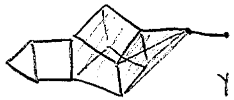
Lecture 16
Oct 1

Def A <u>polyhedral complex</u> $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^d$ such that

(i) $\emptyset \in \mathcal{C}$

(ii) $F \in \mathcal{C} \Rightarrow$ (any face of G) $\in \mathcal{C}$

(iii) $F, G \in \mathcal{C} \Rightarrow F \cap G$ is a face of F and of G.



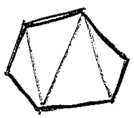
"face to face intersections"

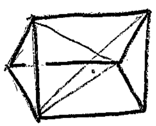
The <u>underlying set</u> is $|\mathcal{C}| = \bigcup_{F \in \mathcal{C}} F$.

Def A <u>subdivision</u> of a polytope P is a polyhedral complex $\mathcal{C}$ with $|\mathcal{C}| = P$

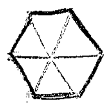
We also demand $V(\mathcal{C}) = V(P)$ (no new vertices)

Ex:





Non-ex:

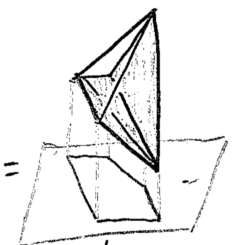


Def A <u>triangulation</u> is a subdivision into simplices.

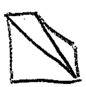
Intuitive: Every P has a triangulation.

One Proof: Constructive!

Illustration:

$P =$  $\subset \mathbb{R}^2 \rightarrow P' =$ 

look at it "from below"

$\mathcal{C} =$ 

Regular subdivisions:

Let $P = \text{conv}(V) \subset \mathbb{R}^d$

Choose $h : V \to \mathbb{R}$

$\Rightarrow$ let $P' = \text{conv}\left\{ \begin{bmatrix} v \\ h(v) \end{bmatrix} : v \in V \right\} \subset \mathbb{R}^{d+1}$

Say a face F of P' is <u>lower</u> if $F = (P')_c$ for some $c \in (\mathbb{R}^{d+1})^*$ with $c_{d+1} < 0$.

Let $\pi : P' \to P$

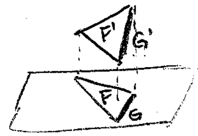
$\begin{bmatrix} x \\ x_{d+1} \end{bmatrix} \mapsto x$

Prop $\{\emptyset\} \cup \{\pi(F) : F$ lower face of $P'\}$ is a subdiv. of P

If h is generic, then it is a triangulation

Pf We consider the generic case, where P' is simplicial, so all $\pi(F)$ are simplices. (Note: π affine isomorphism on each lower face of P', so $\mathcal{C}$ and lower(P') have same face poset)

(i) $\emptyset \in \mathcal{C}$ ✓ (ii) Sup $F \in \mathcal{C}$, G face of F.



Let $F = \pi(F')$

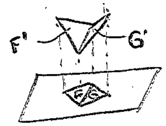
Let G' be the face of F such that $G = \pi(G')$

Since $F' = (P')_c \quad c_{d+1} < 0$

$G' = (P')_{c + \varepsilon c'} \quad (c + \varepsilon c')_{d+1} < 0 \Rightarrow G'$ lower ✓

(iii) Let $F = \pi(F')$, $G = \pi(G')$

Then $F \cap G = \pi(F' \cap G')$ and this property is inherited from P' ✓



Finally, $|\mathcal{C}| = P$:

Let $p \in \text{relint}\, P$. Let $\pi^{-1}(p) = [p_1, p_2]$

Let $F = P_c$ be a face containing p_1.

Then $c \cdot p_1 > c \cdot p_2$

$c \cdot (p_1 - p_2) \geq 0 \Rightarrow c_{d+1} < 0$

So F is lower and $p \in \pi(F)$. ∎