Theorem (Perles' Conj. 1970, Blind-Mani 1987, Kalai 88)
If P is simple, then $G(P)$ determines P combinatorially.

Lecture 15
Sep 29

Pf

$\mathcal{O}(P) = \{\text{acyclic orientations of } G(P)\}$

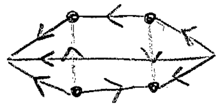
a.o. $\longleftrightarrow$ poset

$u \to \cdots \to v \;\mapsto\; u < v$

Recall: $c \in \mathbb{R}^d$ generic $\longrightarrow$ orientation $O_c \in \mathcal{O}(P)$

O_c has the property that, for any $U \subset V$, the restriction $O_c|_U$ has a unique sink.

Say $O \in \mathcal{O}(P)$ is good if $O|_{G(F)}$ has a unique sink for all faces F of P. (otherwise, bad)

Ex:  is bad

How to tell good from bad?

Let $f^0 = \sum_{v \in V(P)} 2^{\text{indeg}(v)}$

Then, for any $O \in \mathcal{O}(P)$

$$\begin{pmatrix}\#\text{ nonempty}\\ \text{faces}\end{pmatrix} \le \left(\#\text{ pairs } (F,v) : \begin{array}{l} F \text{ face of } P \\ v \text{ sink vertex of } F\end{array}\right)$$

$$= \sum_{v \in V(P)} (\#\text{ faces for which } v \text{ is a sink})$$

$$= \sum_{v \in V(P)} 2^{\text{indeg}(v)} = f^0$$

with equality if and only if O is good.

So: (good orientations O) = (those with max f^0)

A graph is k-regular if all vertices have degree k.

How to tell faces from non-faces?

Claim. Let $U \subset V(P)$

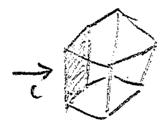
$$\begin{pmatrix} U \text{ is the} \\ \text{vertex set of} \\ \text{a face of } P\end{pmatrix} \Leftrightarrow$$

- $G(P)|_U$ is k-regular
- $\exists$ a good orientation O for which U is a downset ($u \to v$ ($u<v$), $v \in U \Rightarrow u \in U$)

Pf.

$\Rightarrow$: Sup U forms a face. Simple $\Rightarrow$ k-regular

Let it minimize a linear function c



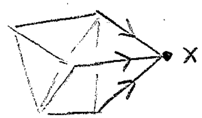
For tiny ε, the linear function $c + d\varepsilon$

- still takes smallest values at U
- induces a good orientation $O_{c+d\varepsilon}$ as desired

$\Leftarrow$: Let $U \subset V(P)$ be as in RHS. $O \in \mathcal{O}(P)$

The orientation $O|_U$ is acyclic; let x be a sink.

It has in-degree k. $O|_U$



Let F be the k-face containing these k edges. O is good so x is the unique sink in $O|_{G(F)}$ and all verts of F are $\le x$. Since U is a downset, $V(F) \subseteq U$.

But $G|_{V(F)}$ and $G|_U$ are k-regular $\Rightarrow U = V(F)$.

So from this criterion we recover $L(P)$ from $G(P)$. ∎

Ⓠ Efficient algorithms??