Balinski's Theorem

A graph is connected if there is a path between any two vertices.

It is d-connected if, after removing any $\leq d-1$ vertices (and all incident edges) it is still connected.

Theorem (Balinski)
P d-polytope $\Rightarrow G(P)$ d-connected

(In particular, deg(vertex) $\geq d$.)

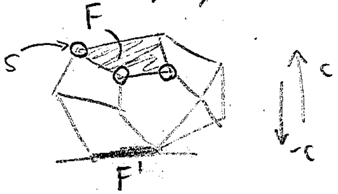
Pf Let $V = \{\text{vertices of } P\}$
$S \subset V \quad |S| \leq d-1$.

Claim: $G(P) \setminus S$ is connected

Pf. We induct on d. $d=1$: ✓ Let $L = \text{span } S$

Case 1: L doesn't intersect int P.

Then $S \subset F$ for some face $F \subsetneq P$,
Say $F = P_c$.
Let $F' = P_{-c}$



By linear programming, every vertex has a c-decreasing path to F' (that doesn't use S). By induction, $G(F')$ is connected.

So $G(P) \setminus S$ is connected.

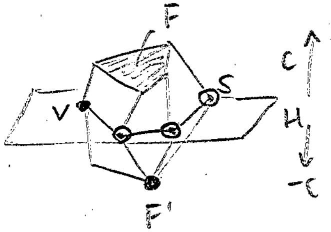
Case 2 L intersects int P.

Let H be a hyperplane containing S and at least one vertex $v \in V \setminus S$. ($\leftarrow \leq d$ points)

Say $H = \{x \in P : cx = c_0\}$.

This is not a facet!

Let $F = P_c$, $F' = P_{-c}$



Let $P^+ = \{x \in P : cx \geq c_0\}$
$P^- = \{x \in P : cx \leq c_0\}$

Then:
- any vertex of P^+ has a c-increasing path to F (avoiding S)
- $G(F)$ connected by induction.

So $G(P^+) \setminus S$ connected

Also $G(P^-) \setminus S$ connected

Also v is in both $\Rightarrow G(P) \setminus S$ connected ∎

Note:

- Steinitz: (graphs of 3-polytopes) $\Leftrightarrow$ (simple, planar, 3-connected graphs)
  simple: no ∞, no ⊖
  planar: drawable in the plane with no edge intersection

  In higher-dim, ???

- This says something about the "dimension" of a $G(P)$. But $G(P)$ can be "dimensionally ambiguous", e.g., $G(P) = K_n$ is possible for $\dim P = n, 2n, 2n+1, 2n+2, \ldots$
  This is not well-understood.
  Open: Is $G(d\text{-cube})$ dim. ambig.? Is $G(C_5)$?