How good is linear programming?

That is related to how long these paths get.

P polytope

$\delta(P)$ = diameter of P

= largest between two vertices of P

let $\Delta(d,n)$ = max diameter of a d-dim polytope with n facets.

How large is $\Delta(d,n)$? Is it polynomial in n and/or d?

Hirsch Conjecture (1957)

$\Delta(d,n) \leq n-d$

Theorem (Santos, June 2010)

The Hirsch conjecture is false! There is a 43-polytope P with 86 facets such that $\delta(P) \geq 44$.

Still, understanding $\Delta(d,n)$ is very much open.

Open. Is $\Delta(d,n) \leq$ polynomial in n,d?

How did he do this?

Lecture 14
Sep 27

Step 1 (Klee–Walkup)

$$\begin{pmatrix}\Delta(d,n) \leq n-d \\ \text{all } n,d\end{pmatrix} \longleftrightarrow \begin{pmatrix}\Delta(d,2d) \leq d \\ \text{for all } d\end{pmatrix}$$

Hirsch Conjecture $\qquad$ d-step Conjecture

Sketch of Proof: (Details in HW)

(HW) ∘ If $n<2d$, $\Delta(d,n) \leq \Delta(d-1,n-1)$

d-step

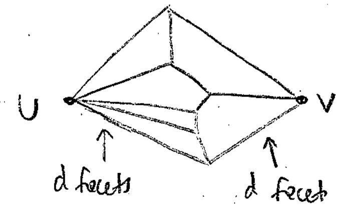
$\Rightarrow \Delta(d,n) \leq \cdots \leq \Delta(n-d, 2(n-d)) \overset{?}{\leq} n-d$

(HW) ∘ If $n>2d$, $\Delta(d,n) \leq \Delta(d+1,n+1)$

d-step

$\Delta(d,n) \leq \Delta(d+1,n+1) \leq \ldots \leq \Delta(n-d, 2(n-d)) \overset{?}{\leq} n-d$ ◼

Ok, but dim = d, facets = $2d$ $\longrightarrow$



If u,v share a facet, restrict to dim $d-1$, induct

If they don't, this is a "spindle". Diameter $\delta(P) = d(u,v)$

Step 2.
If P is a spindle of dim d, $n>2d$ facets, diam. $\delta > d$
there is a spindle of dim $d+1$, $n+1$ facets, diam $\delta+1 > d+1$

Step 3.
There is a spindle of dim 5, 48 facets, diam 6

→ 6, 49 $\qquad$ 7

→ 7, 50 $\qquad$ 8

⋮

Thm.
There is a spindle of dim 43, 86 facets, diam 44

Note A trick allows us to see a 5-spindle on a 3-sphere. So "proof is in 3-D".