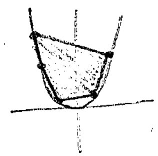
Parenthesis: $\boxed{\text{The Cyclic Polytope}}$

(A polytope with beautiful -and extreme- face structure.)

Let $n, d \in \mathbb{N}$. The <u>moment curve</u> in $\mathbb{R}^d$ is

$$x : \mathbb{R} \to \mathbb{R}^d \qquad t \mapsto x(t) = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$$



The <u>cyclic polytope</u> is

$$C_d(t_1, \dots, t_n) = \operatorname{conv}(x(t_1), \dots, x(t_n)) \qquad (n > d)$$

We call it $C_d(n)$ since we'll show combin. depends only on $d, n$

<u>Lemma</u>. (Vandermonde)

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_d \\ \vdots & \vdots & & \vdots \\ a_0^d & a_1^d & \cdots & a_d^d \end{bmatrix} = \prod_{0 \le i < j \le d} (a_j - a_i)$$

<u>Pf</u>
- LHS is polynomial in the $a_i$'s.
- If $a_i = a_j$, $\det = 0 \Rightarrow (a_j - a_i)$ is a factor
- $\deg \text{LHS} = 1 + \dots + d = \binom{d+1}{2} = \deg \text{RHS}$, so $\text{LHS} = c \cdot \text{RHS}$ constant
- $a_1 a_2^2 \cdots a_d^d$ has coeff 1 on both sides, so LHS = RHS. ∎

<u>Corollary</u>:
- Any $d+1$ vertices of $C_d(t_1 \dots t_n)$ are aff. indep.
- $C_d(t_1 \cdots t_n)$ is simplicial

Observe: For $S \subset [n]$, $|S| = d$, let $H_S$ be the hyperplane through $x(t_{s_1}), \dots, x(t_{s_d})$ $\quad S = \{s_1, \dots, s_d\}$

Its equation is

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x & x(t_{s_1}) & \cdots & x(t_{s_d}) \end{bmatrix} = F_S(x) = 0.$$



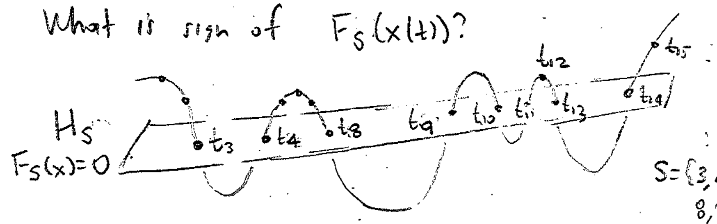
<u>Theorem</u>: $C_d(t_1 \dots t_n)$ is simplicial. For $S \subset [n]$, $|S| = d$,

$$\begin{pmatrix} \{x(t_s) : s \in S\} \\ \text{form a facet} \end{pmatrix} \iff \begin{pmatrix} \text{For all } i < j \text{ not in } S, \\ \#\{k : k \in S,\ i < k < j\} \text{ is even} \end{pmatrix}$$

"no inner blocks are odd"

<u>Pf</u>. Consider the hyperplane $H_S$.
It is a facet $\iff F_S(x(t_i))$ has same sign for all $i \notin S$.

What is sign of $F_S(x(t))$?




$S = \{3, 4, 8, 9, 10, 11, 13, 14\}$

As $t$ goes from $-\infty$ to $\infty$, this sign changes at each $t_s$ $(s \in S)$. So we need it to change an even # of times between $i \notin S$ and $j \notin S$. ∎

<u>Corollary</u>. Combin. type of $C_d(t_1 \dots t_n)$ depends only on $d, n$.

<u>Pf</u> Have V-desc. of facets. Faces = $\cap$ of facets. ∎

<u>Ex</u> $C_3(5)$: facets: 123, 125, 134, 145, 235, 345

$\Rightarrow$ edges: 12, 13, 14, 15, 23, 25, 35, 45 $\quad$ vertices: 1, 2, 3, 4, 5

<u>Corollary</u>. For $|S| \subset [n]$, $|S| = d - k$,

$$\begin{pmatrix} \{x(t_s) : s \in S\} \\ \text{form a face} \end{pmatrix} \iff \begin{pmatrix} \text{at most } k \text{ inner} \\ \text{blocks of } S \text{ are odd} \end{pmatrix}$$

<u>Corollary</u>. $C_d(n)$ is "$\lfloor d/2 \rfloor$-neighborly": Any $\lfloor d/2 \rfloor$ vertices form a face.

<u>Upper Bound Theorem</u> (McMullen)
<u>P</u> $d$-polytope with $n$ vertices $\Rightarrow f_k(P) \le f_k(C_d(n))$ (all $k$).