(c) $P=\{x: Ax\le 1\}$

$P^\Delta=\{a: \text{if } Ax\le 1 \text{ then } ax\le 1\}$

- Let $a\in P^\Delta$. Goal: $a\in \operatorname{conv}(\text{rows } A)$. By Farkas IV, either
  - $Ax\le 1$ is empty (not true: $P\neq\emptyset$) or
  - A (≥ 0)-comb. of $Ax\le 1$ is $ax\le a_0$ for $a_0\le 1$:

    $c'A=a \qquad c'\cdot\mathbb{1}\le 1 \qquad c'\ge 0$

    What I really need is

    $cA=a \qquad c\cdot\mathbb{1}=1 \qquad c\ge 0$

    so it suffices to find $d=c-c'$ with

    $dA=0 \qquad d\cdot\mathbb{1}=\underbrace{1-c'\cdot\mathbb{1}}_{>0} \qquad d\ge 0$

    or, by scaling,

    $c''A=0 \qquad c''\cdot\mathbb{1}=1 \qquad c''\ge 0$

    Farkas II says I can find it unless there exists $x, -y$: $\begin{pmatrix}A\\ \mathbb{1}\end{pmatrix}(x-y)\le\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}(x-y)>0$

    i.e., $Ax\le y\cdot\mathbb{1}$, $y<0$.

    But if such an x existed, then $\lambda x\in P$ (all $\lambda>0$) contradicting that P is bounded.
- $\operatorname{conv}(\text{rows}(A))\subseteq P^\Delta$ is a straightforward computation. ■

Theorem Let $P=\operatorname{conv} V=P(A,1)$ be a polytope and $F=\operatorname{conv} V'=\{x: A''x\le 1, A'x=1\}$ a face where $V=V'\sqcup V''$, $A=A'\sqcup A''$. Then $P^\Delta=\operatorname{conv}(\text{rows } A)=\{a: aV\le 1\}$ has a "dual face"

$F^\Delta=\operatorname{conv}(\text{rows } A')=\{a: aV''\le 1, aV'=1\}$

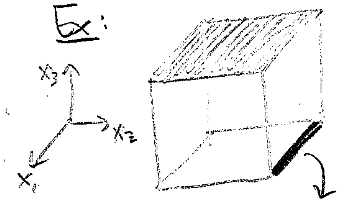
and every face of P^Δ arises in this way.

Also, $F^\Delta=\{c: cx\le 1 \text{ for } x\in P,\ cx=1 \text{ for } x\in F\}$

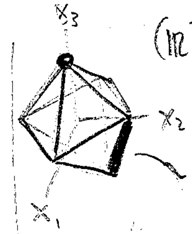
Corollary The face lattices of P and P^Δ are "opposites": $L(P^\Delta)=L(P)^{op}$

Pf. Easy with what we've done.

Ex:



$\mathbb{R}^3$ (col. vecs.)



$(\mathbb{R}^3)^*$ (row vecs.)

conv rows $\begin{bmatrix}1&0&0\\-1&0&0\\0&1&0\\0&-1&0\\0&0&1\\0&0&-1\end{bmatrix}$

$$\operatorname{conv}\underbrace{\begin{bmatrix}1&1&1&1&-1&-1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\1&-1&1&-1&1&-1&1&-1\end{bmatrix}}_{V'}$$

$$=\left\{x: \underbrace{\begin{bmatrix}1&0&0\\-1&0&0\\0&1&0\\0&-1&0\\0&0&1\\0&0&-1\end{bmatrix}}_{A'}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\le\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\right\}$$

$$\left\{x: [x_1\,x_2\,x_3]\begin{bmatrix}1&1&1&1&-1&-1&-1&-1\\1&1&-1&-1&1&1&-1&-1\\1&-1&1&-1&1&-1&1&-1\end{bmatrix}\le[1\,1\,1\,1\,1\,1\,1\,1]\right\}$$

Note Metric of P^Δ depends on metric of P. Combin. of P^Δ doesn't (as long as $0\in\operatorname{int}P$)

Simple and Simplicial Polytopes

- A d-polytope is *simplicial* if every facet (or every proper face) is a simplex.

  Equiv: (every interval $[\hat{0},F]$ ($F\neq\hat{1}$) is Boolean)

  (moving vertices a little doesn't change combinatorics)

- A d-polytope is *simple* if every vertex is on d edges

  Equiv: (every interval $(F,\hat{1})$ ($F\neq\hat{0}$) is Boolean)

  (moving facets a little doesn't change combinatorics.)

$$\boxed{(P \text{ simple})\iff(P^\Delta \text{ simplicial})}$$