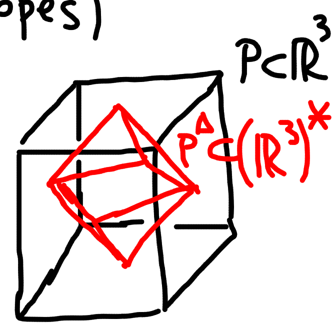
Polar Polytopes (Dual polytopes)

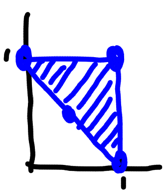
$P \subset \mathbb{R}^d$ polytope (or some other subset)

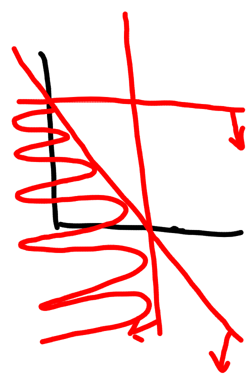


The <u>polar</u> of P is:

$$P^\Delta = \{c \in (\mathbb{R}^d)^* : cx \le 1 \text{, all } x \in P\}$$

Ex.

$P =$ 

$P^\Delta =$ 

V vector space over $\mathbb{R}$

$\downarrow$

$V^* = \{$linear functions $f : V \to \mathbb{R}\}$ "functional"

$\cong V$

Theorem

P, Q polytopes

$$P^\Delta = \{c \in (\mathbb{R}^d)^* : cx \leq 1, \text{ all } x \in P\}$$

✓ (a) $P \subseteq Q \Rightarrow P^\Delta \supseteq Q^\Delta$

Ex. (b) $P \subseteq P^{\Delta\Delta}$

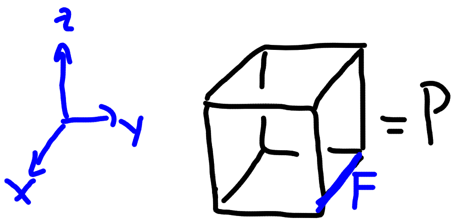
(c) If $0 \in P$ then $P = P^{\Delta\Delta}$

(d) If $(0 \in \operatorname{int} P)$ and $P = \operatorname{conv}(V)$ then

$$P^\Delta = \{a : av \leq 1 \quad \text{for all } v \in V\}$$

(e) If $(0 \in \operatorname{int} P)$ and $P = P(A, \mathbb{1})$ then

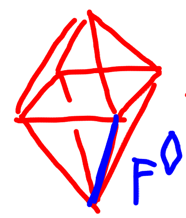
$$P^\Delta = \{cA : c \geq 0,\ c \cdot \mathbb{1} = 1\} = \operatorname{conv}(\operatorname{rows}(A))$$



$= P$

$$\operatorname{conv}\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



$= P^{\Delta}$

$$\operatorname{conv}\left(\text{rows}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\right)$$

$$[x\ y\ z]\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \le [1\,1\,1\,1\,1\,1\,1\,1]$$

(c) If $\underline{0 \in P}$ then $P = P^{\Delta\Delta}$

Need: $P^{\Delta\Delta} \subset P$

Sup $q \in P^{\Delta\Delta}$, $q \notin P$

Let $c \cdot x = c_0$ sep. q from P:

$$\begin{cases} c \cdot q > c_0 \\ c \cdot p < c_0 \text{ for } p \in P \xrightarrow{0 \in P} 0 < c_0 \end{cases}$$

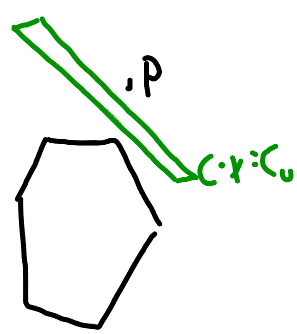
$$\begin{cases} c/c_0 \cdot q > 1 \\ c/c_0 \cdot p < 1 \text{ for } p \in P \Rightarrow \frac{c}{c_0} \in P^{\Delta} \end{cases} \Bigg) \; \frac{c}{c_0} \cdot q \leq 1$$

$q \in P^{\Delta\Delta}$

$P^{\Delta} = \{ c : cx \leq 1 \;\; \forall x \in P \}$

$P^{\Delta\Delta} = \{ y : cy \leq 1 \;\; c \in P^{\Delta} \}$

$= \{ y :$ if $cx \leq 1$ for $x \in P$ then $cy \leq 1 \}$



(d) If ($0 \in \text{int } P$ and) $P = \text{conv}(V)$

$$P^\Delta = \{a : a \cdot v \leq 1 \quad \forall v \in V\}$$

Pf

$\subseteq$: $P^\Delta = \{a : a \cdot p \leq 1 \quad \forall p \in P\}$ ✓

$\supseteq$: Let a be such that $a \cdot v \leq 1 \quad \forall v \in V$

Sup $a \cdot p > 1$ for some $p \in P$.

The linear functional $a \cdot \_$ is max at face P_a. Let v be a vertex of P_a

$$a \cdot v \geq a \cdot p > 1$$

(e) If ($0 \in \operatorname{int} P$ and) $P = P(A, \mathbb{1})$

$$P^\Delta = \{cA : c \geq 0,\ c \cdot \mathbb{1} = 1\} = \operatorname{conv}(\operatorname{rows}(A))$$

$\supseteq$: easy.

$\subseteq$: Let $a \in P^\Delta$. Either

- ~~$P = \emptyset$~~ $(0 \in P)$ or
- $ax \leq 1$ is a ≥ 0 combin of $Ax \leq \mathbb{1}$

$P^\Delta = \{a : x \in P \Rightarrow a \cdot x \leq 1\}$

$= \{a : Ax \leq \mathbb{1} \Rightarrow ax \leq 1\}$

Farkas IV

If an ineq. holds for P then either

- that ineq is a >0 comb of the ineqs of P

or

- $P = \emptyset$

$ax \leq 1$ is a combin. of $Ax \leq \mathbb{1}$

$\Rightarrow \exists\, c'$ such that $c'A = a$, $c' \cdot \mathbb{1} = e \leq 1$, $c' \geq 0$

We need: $\exists\, c$ such that $cA = a$, $c \cdot \mathbb{1} = 1$, $c \geq 0$

(Assume $e<1$)

Sufficient: $\exists\, d$ such that $dA = 0$, $d \cdot \mathbb{1} = 1$, $d \geq 0$

$$d = \frac{c - c'}{1-e}$$

Need: $d \begin{bmatrix} A \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $d \geq 0$

Farkas II: If d didn't exist, there must be $(x\ y)$ such that

$$\begin{bmatrix} A \\ \mathbb{1} \end{bmatrix} [x\ y] = 0 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} [x\ y] < 0$$

$$Ax = \mathbb{1} y \qquad y < 0 \Rightarrow x \in P$$

$$\Rightarrow \lambda x \in P \quad \lambda > 0$$