Lecture 7

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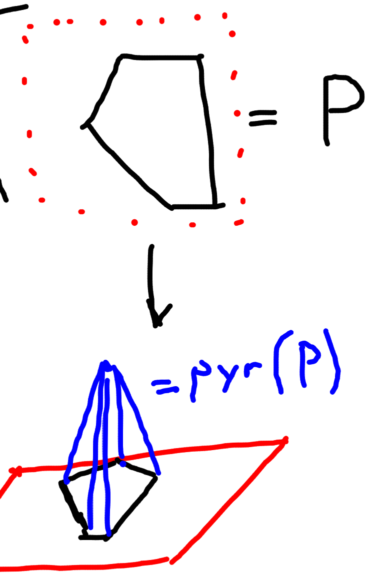
Two more constructions

① Pyramids

$P \subset \mathbb{R}^d$ polytope

Embed $\mathbb{R}^d$ as $\{x_{d+1}=0\} \hookrightarrow \mathbb{R}^{d+1}$

$$\mathrm{Pyr}(P) = \mathrm{conv}\left(\left\{\begin{pmatrix} p \\ 0 \end{pmatrix} : p \in P\right\} \cup \left\{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right\}\right)$$



Face structure of $\mathrm{Pyr}(P)$ is determined by the face structure of P. (HW)

② Vertex Figures

$P \subset \mathbb{R}^d$

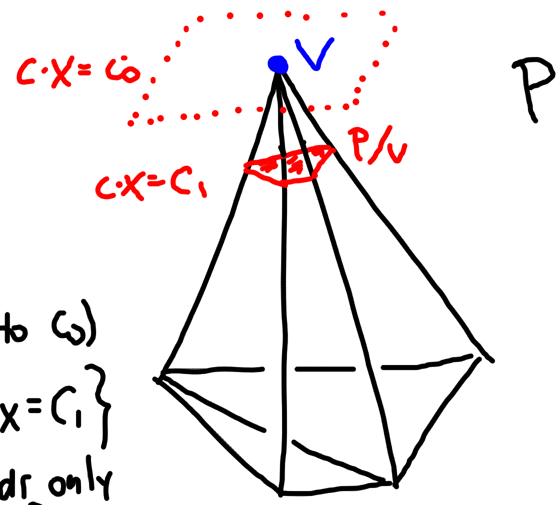
v vertex, say $v = P_c$

$c \cdot v = c_0$

$c \cdot p \leq c_0 \quad p \in P$

Choose $c_1 < c_0$ (but close to c_0)

$P/v = P \cap \{x : c \cdot x = c_1\}$



Combin type of P/v depends only on the combin type of P, v.

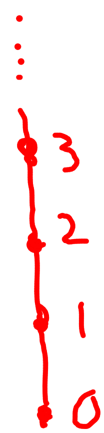
$$\begin{pmatrix} (k-1)\text{-faces} \\ \text{of } P/v \end{pmatrix} \longleftrightarrow \begin{pmatrix} k\text{-faces of } P \\ \text{containing } v \end{pmatrix} \quad (\text{Ex.})$$

THE FACE LATTICE

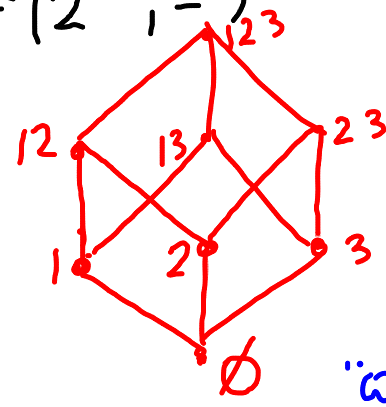
<u>Def</u> A <u>poset</u> (partially ordered set) $(P, \leq)$ is a set P equipped with a binary relation $\leq$ such that:

- $x \leq x$ $\quad \forall x \in P$
- $x \leq y,\ y \leq z \Rightarrow x \leq z$ $\quad \forall x, y, z \in P$
- $x \leq y,\ y \leq x \Rightarrow x = y$

<u>Ex.</u> $(\mathbb{N}, \leq)$



$B_n = (2^{[n]}, \subseteq)$ "Boolean poset"



<u>Hasse diag</u>
- element

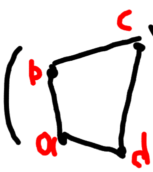
$u < v$

$\nexists w$ s.t. $u < w < v$

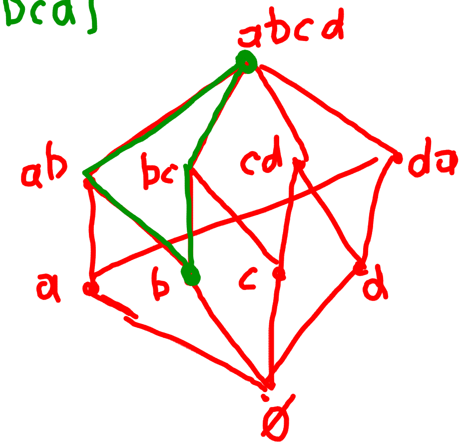
"cover rels"

P polytope

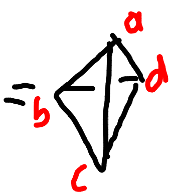
$L(P)$ = "face poset" = $(\{\text{faces of } P\}, \subseteq)$

Ex. $L\Big($  $\Big)$

[b, abcd]



$L(\Delta_{d-1}) = B_d$ (Ex.)

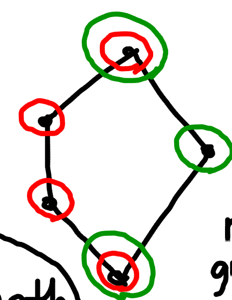
$\Delta_3 =$ 

P, Q are <u>combin. isom.</u> if $L(P) \cong L(Q)$

P poset

chain: $\overbrace{p_1 < p_2 < \cdots < p_k}^{\text{length } k-1}$

P graded if for any $a < b$, all maximal chains from a to b have same length (i.e., P has "levels" or "ranks")



not graded

B_n graded

$rk(S) = |S|$

P is a <u>lattice</u> if

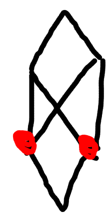
- any $x, y \in P$ have a least upper bound $a \vee b$ "join"
  and a greatest lower bound $a \wedge b$ "meet"

<u>Ex</u>

B_n is a lattice

$A \wedge B = A \cap B$

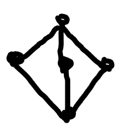
$A \vee B = A \cup B$

 is not a lattice

<u>Note:</u>
Lattices have a maximum $\hat{1}$
+ a minimum $\hat{0}$

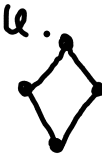
<ins>Theorem</ins> P polytope

(a) $L(P)$ is a lattice, graded by $rk(F) = \dim(F)+1$



(b) Every interval $[F, G]$ is also a face lattice.

(c) Every interval of height 2 is a diamond



(d) The "opposite poset" $L(P)^{op}$ is also a face lattice.

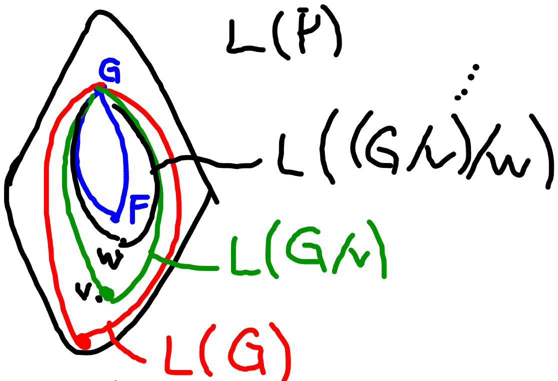
(a) $F \wedge G = F \cap G$

$F \vee G = \bigwedge$ (upper bounds of F, G)

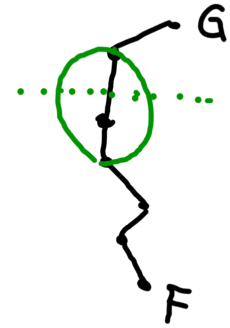
$\left(\begin{array}{l} P \text{ poset} \\ \text{has } \hat{0}, \hat{1} \\ \text{has } \wedge \end{array}\right) \Rightarrow (\text{has } \vee)$

Ex

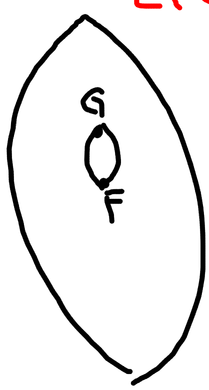
b)



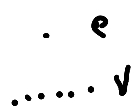
$\rightarrow$ a) Graded



c)



$[F,G] = L(\bullet\!-\!\bullet) = \diamond$



d) $L(P)^{op} \cong \overset{\emptyset}{L}(P^{\Delta})$

polar pol. of P
(next time)