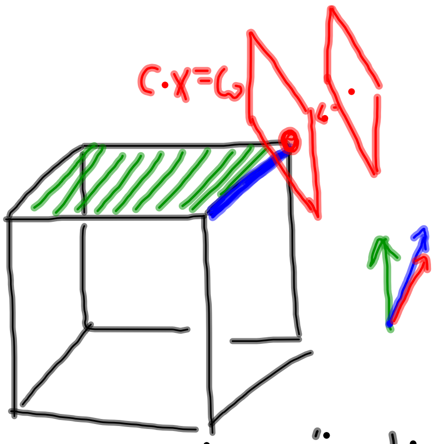
FACES OF POLYTOPES

$P \subset \mathbb{R}^d$ polytope, $c \in \mathbb{R}^d$

The <u>face</u> of P in direction c is

$$P_c = \{x \in P \mid c \cdot x \text{ is max}\}$$



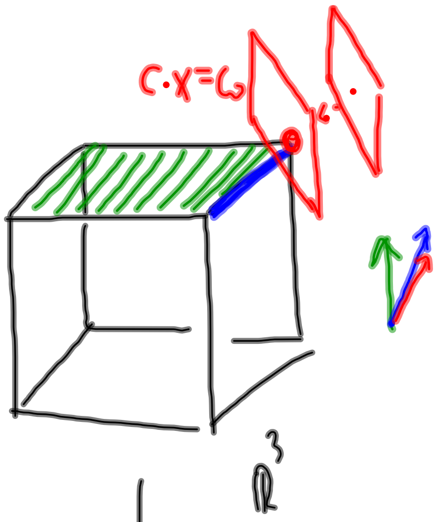
Note: P is bounded $\longrightarrow$ say

$$P_c = \{x \in P \mid c \cdot x = c_0\}$$

$c \cdot x \leq \underset{\min}{c_0}$ is the "smallest" valid ineq for P

dim P_c = ?

Remarks: affine spaces

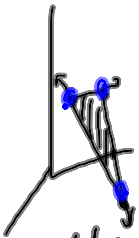


An affine subspace of $\mathbb{R}^d$ is:

- $\{x\in\mathbb{R}^d : Ax=b\}$
- translate of a (vector) subspace
- affine span of a set $V\subset\mathbb{R}^d$

$$\mathrm{aff}(V)=\left\{\lambda_1 v_1+\dots+\lambda_r v_r : v_i\in V,\ \sum\lambda_i=1\right\}$$

Say $V=\{v_1,\dots,v_k\}$ is aff. indep. if no v_i is an affine comb. of $V-v_i$.



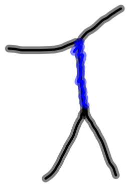
$\dim V=(\text{size of largest aff indep subset})-1$

$$(v_1\dots v_k \text{ aff indep}) \iff \left(\begin{pmatrix}1\\ v_1\end{pmatrix}\dots\begin{pmatrix}1\\ v_k\end{pmatrix} \text{ lin. indep}\right)$$

The $\underline{\dim}$ of a face F of P is $\dim(\operatorname{aff}(F))$.

dim 0 = verts.   codim 1 = facets

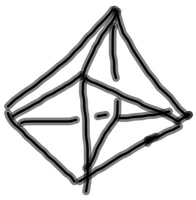
dim 1 = edges   codim 2 = ridges



The f-vector of P is $f(P) = (\underset{\overset{\|}{1}}{f_{-1}}, f_0, \dots, f_d)$

where $f_i :=$ # of i-faces of P

The f-polynomial of P is $f_0 x^0 + \dots + f_d x^d = f_P(x)$

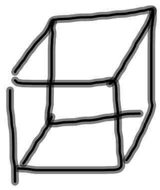


$$f_P(x) = 6 + 12x + 8x^2 + x^3$$

⚠ other authors have slightly diff. conventions

<u>Ex</u> $C_d = \operatorname{conv}(\{-1,1\}^d) = \{x : -1 \le x_i \le 1\}$

C_3



Goal: compute faces.

Let $v \in \mathbb{R}^d$, compute $(C_d)_v$

$$(C_d)_v = \{x \in [-1,1]^d : v_1x_1 + \cdots + v_dx_d \max\}$$

If $v_i > 0$, $v_ix_i \le v_i$ $(x_i = 1)$

$v_i < 0$, $v_ix_i \le -v_i$ $(x_i = -1)$

$v_i = 0$, $v_ix_i = 0$ (any x_i)

Say $v = (>0, <0, 0, >0, 0, 0, <0)$

$$(C_7)_v = \{(1, -1, *, 1, *, *, -1)\} \cong C_3$$

$[-1,1]$

$(\text{faces of } C_d) \xleftrightarrow{\text{bij}}$ sign patterns $\{(+,-,0)\}^d$

dim face $\longleftrightarrow$ # of 0s

$$f_k(C_d) = \binom{d}{k} 2^{d-k}$$

pos. of 0s

other pos. + or −

$$f_{C_d}(x) = \sum_{k=0}^{d} f_k x^k = \sum_{k=0}^{d} \binom{d}{k} 2^{d-k} x^k$$

$$= (x+2)^d$$

There are several "obvious" things to prove.

$\boxed{\underline{\text{Prop}}\ P \text{ polytope} \Rightarrow P = \operatorname{conv}(\operatorname{vert}(P))}$ (Ex.)

Pf let $P = \operatorname{conv}(V)$

If $v \in V$ is s.t. $v \in \operatorname{conv}(V - v) \Rightarrow \operatorname{conv}(V) = \operatorname{conv}(V - v)$

Keep elim. all superfluous members of V until it's no longer possible. Call that set W

Claim: $W = \operatorname{vert}(P)$

$\supseteq$: let $v \in \operatorname{vert} P$, assume $v \notin W \Rightarrow v \in \operatorname{conv}(W - v)$

Write $v = \lambda_1 w_1 + \dots + \lambda_k w_k$ $\quad w_i \in W - v$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$

Assume $c \cdot v = c_0$, $c \cdot p < c_0$ $\forall p \in P - v$

$c \cdot v = \lambda_1 (c \cdot w_1) + \dots + \lambda_k (c \cdot w_k)$

$c_0 < c_0 (\lambda_1 + \dots + \lambda_k) = c_0$

$\subseteq$: Let $w \in W$ $\quad w \notin \operatorname{conv}(\underbrace{W - w}_{W'})$

$w \notin \operatorname{conv} W' \Rightarrow \nexists\, t \geq 0 : w = W't,\ 1 = \mathbb{1}t$

$\Rightarrow \nexists\, t : \begin{pmatrix} \mathbb{1} \\ W' \end{pmatrix} t = \begin{pmatrix} 1 \\ w \end{pmatrix},\ t \geq 0$

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$\Rightarrow \exists\, a : a\begin{pmatrix} \mathbb{1} \\ W' \end{pmatrix} \geq 0,\ a\begin{pmatrix} 1 \\ w \end{pmatrix} < 0$

Need: $\beta \cdot p < b \cdot w$ $\quad p \in P - w$ (Ex)

$\Rightarrow \exists\, (\beta, -b) : \beta \mathbb{1} - bW' \geq 0,\ \beta - bw < 0$

$\Rightarrow \exists\, \beta, b : bW' \leq (\beta \cdots \beta),\ bw > \beta$

$\Rightarrow w$ maximizes $b \cdot x \Rightarrow w \in \operatorname{vert} P$. ∎

Other "obvious" facts about polytopes + their faces

Prop P = polytope $V = \mathrm{vert}(P)$

(i) Any face F of P is a polytope, $\mathrm{vert}(F) = \mathrm{vert}(P) \cap F$

(ii) F, G faces $\Rightarrow F \cap G$ face

(iii) F face of P, G face of $F \Rightarrow G$ face of P

(iv) F face $\Rightarrow F = P \cap \mathrm{aff}(F)$

(ii) $F = P_b,\ G = P_c \Rightarrow F \cap G = P_{b+c}$

Ex

(iii) $F = P_b,\ G = F_c \Rightarrow G = P_{b+\varepsilon \cdot c}$ (ε small)

Ex