Problem 6:

Let e_i,f_j be the standard unit vectors in $\mathbb{R}^m, \mathbb{R}^n$ (resp.), $v_{ij} = e_i \times f_j$, and $\Delta_{m-1} \times \Delta_{n-1} = conv\{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Let $\Gamma :=$ { staircase from $(1,1)$ to $(m,n)\}$, so Γ has $\binom{m+n-2}{m-1}$. For each $S \in \Gamma$, define $P_S := conv\{v_{ij} : (i,j) \in S\}$. Lets prove that $\{P_S : S \in \Gamma\}$ is a triangulation of $\Delta_{m-1} \times \Delta_{n-1}$:

i)**Lets prove that** P_S is a simplex for all $S \in \Gamma$:

To prove this is sufficient to show that the $m+n-1$ vertices of P_S are affinely independent (i.e they doesn't lie in a $m+n-3$-dimensional affine space). Name the vertices of P_S, $w_1, w_2, ..., w_{m+n-1}$, according to the order the appear in the staircase, so $w_1 = v_{11}$ and $w_{m+n-1} = v_{mn}$. Suppouse we have $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{m+n-1} w_{m+n-1} = 0$ for some λ's such that $\lambda_1 + \lambda_2 + ... + \lambda_{m+n-1} = 1$. Let k be the greatest index such that $\lambda_k \neq 0$, then we can write $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{k-1} w_{k-1} = -\lambda_k w_k$. If we write $w_1 = v_{a_1 b_1}, w_2 = v_{a_2 b_2}, ..., w_k = v_{a_k b_k}$,and remembering that the points $w_1, w_2, ..., w_k$ are ordered according a staircase, we can conclude that one of the following conditions must hold: $a_k > a_i$ for all $i < k$, or $b_k > b_i$ for all $i < k$. WLOG assume $a_k > a_i$ for all $i < k$. Then the a_k-th component of the vector $w_k = v_{a_k b_k}$ is 1, while the a_k-th component of the vectors $w_i = v_{a_i b_i}$ is 0 for all $i < k$.Therefore we cannot have the equality $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{k-1} w_{k-1} = -\lambda_k w_k$, where $\lambda_k \neq 0$. This implies that $\lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_{m+n-1} w_{m+n-1} = 0$ only holds when $\lambda_i = 0$ for all i, so $w_1, w_2, ..., w_{m+n-1}$ are affinely independent.

ii) **Lets prove that** $\cup_{\{S\in\Gamma\}}P_S = \Delta_{m-1}\times\Delta_{n-1}$:
For any $x\in\Delta_{m-1}\times\Delta_{n-1}$ write $x=(\alpha_1,\alpha_2,...,\alpha_m,\beta_1,\beta_2,...,\beta_n)$. Since $(\alpha_1,\alpha_2,...,\alpha_m)\in\Delta_{m-1}$, we get that $\alpha_i\geq 0$ for all i, and $\alpha_1+\alpha_2+...+\alpha_m=1$. Similarly, since $(\beta_1,\beta_2,...,\beta_n)\in\Delta_{n-1}$, we get that $\beta_j\geq 0$ for all j, and $\beta_1+\beta_2+...+\beta_n=1$.

Define $A_1=\alpha_1, A_2=\alpha_1+\alpha_2, A_3=\alpha_1+\alpha_2+\alpha_3,..., A_m=\alpha_1+\alpha_2+\alpha_3+...+\alpha_m=1$, and $B_1=\beta_1, B_2=\beta_1+\beta_2, B_3=\beta_1+\beta_2+\beta_3,..., B_n=\beta_1+\beta_2+\beta_3+...+\beta_n=1$. Observe that $0\leq A_1\leq A_2\leq...\leq A_m=1$ and $0\leq B_1\leq B_2\leq...\leq B_n=1$. We can "mix" the previous sequences in a single ordered chain of lenght $m+n$, for instance if $A_1=0, A_2=0.6, A_3=0.8, A_4=1$, $B_1=0.3, B_2=0.5, B_3=1$, we get $A_1\leq B_1\leq B_2\leq A_2\leq A_3\leq B_3\leq A_4$. Since $0\leq A_1\leq A_2\leq ...\leq A_m=1$ and $0\leq B_1\leq B_2\leq...\leq B_m=1$, there are exactly $\binom{m+n-2}{m-1}$ classes of chains (I mean, chains with the identical order of A_i's and B_i's, up two the order of $A_m=B_n=1$ in the last two places of the chain), that are obtained by selecting the $m-1$ positions of $A_1,A_2,...,A_{m-1}$ in the first $m+n-2$ places of the chain. For instance $A_1\leq B_1\leq B_2\leq A_2\leq A_3\leq B_3\leq A_4$ and $A_1\leq B_1\leq A_2\leq B_2\leq A_3\leq B_3\leq A_4$ are **diferent** classes of chains, but $A_1\leq B_1\leq B_2\leq A_2\leq A_3\leq B_3\leq A_4$ and $A_1\leq B_1\leq B_2\leq A_2\leq A_3\leq A_4\leq B_3$ are the **same** class.

Therefore the amount of classes of chains is equal to the number of staicases in Γ. Let see the relation. For a given chain construct a staircase as follows:

0) Start at $(1,1)$.
1) If the first element of the chain is A_1 move to the east, if it is B_1 move to the north.
2) If the k-th element of the chain is of the form A move to the east, if it is of the form B move to the north.

Fix $x\in\Delta_{m-1}\times\Delta_{n-1}$, let C_x be a chain related to x, and let S_x be the staircaes induced by C_x using the previous construction. I claim that $x\in P_{S_x}$:

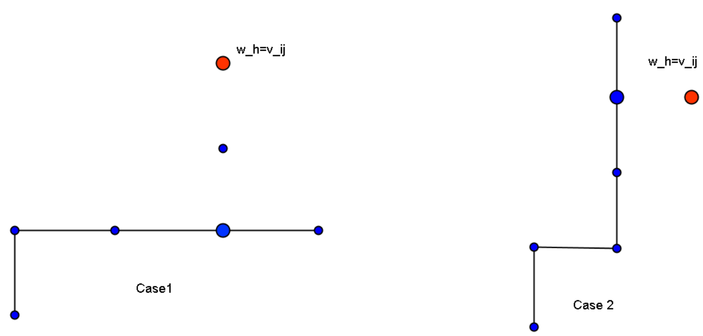
Write the chain C_x associated to x in the form $C_1\leq C_2\leq...\leq C_{m+n}$ (for instance if C_x is the chain $A_1\leq B_1\leq B_2\leq A_2\leq A_3\leq B_3\leq A_4$, then we have $C_1=A_1, C_2=A_2, C_3=B_2,...,C_7=A_4$). Name the vertices of P_{S_x}, $w_1,w_2,...,w_{m+n-1}$, according to the order the appear in the staircase, so $w_1=v_{11}$ and $w_{m+n-1}=v_{mn}$. **Now, we can check that** $x=C_1w_1+(C_2-C_1)w_2+...+(C_{m+n-1}-C_{m+n-2})w_{m+n-1}$. This show that $x\in P_{S_x}$ since $C_1w_1+(C_2-C_1)w_2+...+(C_{m+n-1}-C_{m+n-2})w_{m+n-1}$ is a convex combination of the vertices of P_{S_x}.

For example suppouse $x=(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3)=(0,0.6,0.2,0.2,0.3,0.2,0.5)$ so $A_1=0, A_2=0.6, A_3=0.8, A_4=1$, and $B_1=0.3, B_2=0.5, B_3=1$. Then we can take C_x, the chain associated to x, as $A_1\leq B_1\leq B_2\leq A_2\leq A_3\leq B_3\leq A_4$. Using the construction of the staircase from the chain C_x,we get the following order of movements: east,north,north,east,east. This produces the vertices $v_{1,1},v_{2,1},v_{2,2},v_{2,3},v_{3,3},v_{4,3}$. Writting the chain C_x in the form $C_1\leq C_2\leq...\leq C_7$, observe that $C_1v_{1,1}+(C_2-C_1)v_{2,1}+...+(C_6-C_5)v_{4,3}=$

$$0(1000100)+0.3(0100100)+0.2(0100010)+0.1(0100001)+0.2(0010001)+0.2(0001001)=$$

$$(0,0.6,0.2,0.2,0.3,0.2,0.5)=x$$

iii) **Lets prove that** $P_{S_1} \cap P_{S_2}$ is a face of both of them. Let $\{v_1, v_2, ..., v_k\} = V(P_{S_1}) \cap V(P_{S_2})$, I claim that $P_{S_1} \cap P_{S_2} = conv\{v_1, v_2, ..., v_k\}$. To prove this I will argue by contradiction. Suppouse there exists $q \in (P_{S_1} \cap P_{S_2}) \setminus conv\{v_1, v_2, ..., v_k\}$. Then, there exists $w_h \in V(P_{S_1}) \setminus V(P_{S_2})$ which is component of q, i.e,if we write q as a convex combination of the vertices of P_{S_1}, then the coefficient of w_h, say λ_h is greater than 0 (since P_{S_1} is a simplex, the point q can be written in a unique way as convex combination of the vertices of P_{S_1}). Let $w_h = v_{ij}$. Since $w_h \notin V(P_{S_2})$, there exists $\hat{j} < j$ such that $v_{i,\hat{j}}$ and $v_{i+1,\hat{j}}$ belong to $V(P_{S_2})$ (*Case 1*), **or** there exist $\hat{i} < i$ such that $v_{\hat{i},j}$ and $v_{\hat{i},j+1}$ belong to $V(P_{S_2})$(*Case 2*) . These two cases are presented below:



a) Case 1: When we write q as a point in P_{S_1} the condition $\lambda_h > 0$ (which is the coefficient of $w_h = v_{ij}$) implies $A_i > B_{j-1}$. On the other hand, when we write q as a point in P_{S_2}, the existence of $v_{i,\hat{j}}$ and $v_{i+1,\hat{j}}$ in $V(P_{S_2})$, with $\hat{j} < j$, implies $A_i \leq B_{\hat{j}} \leq B_{j-1}$. The conditions $A_i > B_{j-1}$ and $A_i \leq B_{j-1}$ lead us to a contradiction. Therefore $(P_{S_1} \cap P_{S_2}) \setminus conv\{v_1, v_2, ..., v_k\} = \emptyset$, so $P_{S_1} \cap P_{S_2} = conv\{v_1, v_2, ..., v_k\}$. Since P_{S_1} and P_{S_2} are simplexes, then $conv\{v_1, v_2, ..., v_k\}$ is a face of both of them.

b)Case 2: This case is analogous to the previous one. It leads to the contradiction ,$B_j > A_{i-1}$ and $B_j \leq A_{i-1}$.