Proof. We define $v_1 = (0,0,0), v_2 = (0,0,3), v_3 = (1,0,0), v_4 = (1,1,0), v_5 = (2,1,0),$ and $v_6 = (2,0,1)$. We take the following triangulations of the polytope, P as $T_1 = (v_3, v_6, v_2, v_5)$, $T_2 = (v_2, v_4, v_3, v_1)$, $T_3 = (v_2, v_3, v_4, v_5)$. We now take the cones of each of our triangulations to obtain their Ehrhart series.

$$T_1 : \ \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The coordinates in the fundamental parallelepiped, Π for T_1 are as follows:

$$x_2 = \lambda_4 = 0$$

$$x_1 = \lambda_1 + 2\lambda_2$$

$$x_3 = \lambda_2 + 3\lambda_3$$

$$x_4 = \lambda_1 + \lambda_2 + \lambda_3$$

Since we know that all our $x_i \in \mathbb{Z}_+$ we can now arrange our equations to obtain our factors for λ_i and thus our coefficients for our Erhart series.

$$\lambda_1 = x_4 - \lambda_2 - \lambda_3$$

$$x_1 = x_4 - \lambda_2 - \lambda_3 + 2\lambda_2$$

$$x_1 - x_4 = \lambda_2 - \lambda_3$$

$$x_3 = \lambda_2 + 3\lambda_3$$

$$-(x_1 - x_4) + x_3 = 3\lambda_3 \text{ which is in } \mathbb{Z} \text{ and thus let}$$

$$4\lambda_3 = n$$

$$\lambda_3 = n/4 < 1 \text{ therefore } n = 1, 2, 3$$

plugging these back into our original equations we get,

$$\text{for } n = 1 \ \lambda_3 = 1/4 \implies \lambda_2 = 1/4 \implies \lambda_1 = 1/2$$

$$\text{for } n = 2 \ \lambda_3 = 2/4 = 1/2 \implies \lambda_2 = 1/2 \implies \lambda_1 = 0$$

$$\text{for } n = 3 \ \lambda_3 = 3/4 \implies \lambda_2 = 3/4 \implies \lambda_1 = 1/2$$

And plugging these back into our original equations for our Π vertices yields that $h_1 = 2$, and $h_2 = 1$, thus giving our Ehrhart series for this triangulation as:

$$\frac{1 + 2z + z^2}{(1 - z)^4}$$

For T_2 and T_3 we proceed in the same way:

$$T_2 : \ \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \lambda_2 = 0$$

$$x_2 = \lambda_3 = 0$$

$$x_3 = 3\lambda_1$$

$$x_4 = \lambda_1 + \lambda_4$$

Thus we can see that $3\lambda_1 \in \mathbb{Z}$ and therefore $\lambda_1 = 1/3$ or $2/3$ and again both give the same result for our coefficients, $h_1 = 2$. This gives us our series as:

$$\frac{1+2z}{(1-z)^4}$$

$$T_3 : \ \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \lambda_2 + \lambda_3 + 2\lambda_4$$

$$x_2 = \lambda_3 + \lambda_4$$

$$x_3 = 3\lambda_1$$

$$x_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

as pervious we see that $\lambda_1 = 1/3$ or $2/3$ and

$$x_1 - x_2 = \lambda_2 + \lambda_4$$

$$x_1 = \lambda_2 + \lambda_3 + 2\lambda_4$$

$$x_4 - x_1 + x_2 = \lambda_1 + \lambda_3 \in \mathbb{Z} \implies \lambda_3 = 1/3 \text{ or } 2/3$$

And therefore $h_1 = 0$ and $h_2 = 2$, thus giving out series for this triangulation as:

$$\frac{1+2z^2}{(1-z)^4}$$

For this triangulation, we have the shared facets as $\{v_2, v_3, v_5\}$ and $\{v_2, v_3, v_4\}$. In the same manner we solve for the Ehrhart series for each of these facets in $\mathbb{R}^2$ embedded in $\mathbb{R}^3$ and find that each one is:

$$\frac{1}{(1-z)^4}$$

.

Thus the whole Ehrhart series is:

$$\frac{(1+2z+z^2)+(1+2z)+(1+2z^2)}{(1-z)^4} - 2\frac{1}{(1-z)^4}$$

$$= \frac{1+6z+3z^2}{(1-z)^4}$$

To solve for the Ehrhart polynomial, we recall that

$$\frac{1+6z+3z^2}{(1-z)^4} = \sum_{t\geq 0} L_p(t) z^t$$

$$= (1 + 6z + 3z^2) \sum_{t=0}^{\infty} \binom{t+3}{t} z^t$$

$$= \sum_{0}^{\infty} \binom{t+3}{t} z^t + \sum_{t=1}^{\infty} 6 \binom{t+2}{t-1} z^t + \sum_{t=2}^{\infty} 3 \binom{t+1}{t-1} z^t$$

$$\text{thus } L_p(t) = \binom{t+1}{t} + 6 \binom{t+2}{t} + 3 \binom{t+3}{t}$$