Proof: $\Rightarrow$ Let $g(z)$ be a polynomial of degree $\leq d$ and $g(1) \neq 0$.

Let $g(z) = a_n z^n + \dots a_1 z + a_0$ and $a_n + a_{n-1} + \dots a_1 + a_0 \neq 0$. Then $\frac{g(z)}{(1-z)^{d+1}} = g(z) \sum_{n \geq 0} \binom{t+d}{t} z^t$

$$\frac{g(z)}{(1-z)^{d+1}} = \sum_{i=0}^{n} \sum_{t \geq 0} a_i \binom{t+d}{t} z^{t+i}$$

$$\frac{g(z)}{(1-z)^{d+1}} = \sum_{i=0}^{n} \sum_{t \geq i} a_i \binom{t-i+d}{t-i} z^{t}$$

By convention $\binom{t-i+d}{t-i} = 0$ when $t < i$, hence we have

$$\frac{g(z)}{(1-z)^{d+1}} = \sum_{i=0}^{n} [\sum_{t \geq 0} a_i \binom{t-i+d}{t-i} z^{t}]$$

Now we can switch the summation which gives us

$$\frac{g(z)}{(1-z)^{d+1}} = \sum_{t \geq 0} [\sum_{i=0}^{n} a_i \binom{t-i+d}{t-i}] z^{t}$$

For any $1 \leq i \leq n$ we have $\binom{t-i+d}{t-i} = \frac{(t-i+d)!}{(t-i)!d!} = \frac{(t-i+d)(t-i+d-1)\dots(t-i+1)}{d!} = \frac{\prod_{k=0}^{d-1}(t-i+d-k)}{d!}$. Hence each is a polynomial of degree d.

$$f(t) = \sum_{i=0}^{n} a_i \binom{t-i+d}{t-i} = \frac{a_0 + a_1 + \dots + a_n}{d!} t^d + b_{d-1} + \dots b_1 t + b_0$$

For some coefficients b_k where $0 \leq k \leq d-1$. Since $g(1) = a_0 + a_1 + \dots + a_n \geq 0$ we are assure the coefficient of t^d is not equal to 0 hence the degree of f is d.

$\Leftarrow$ Let $f(t)$ be a polynomial of degree d.
Let $f(t) = b_d t^d + \ldots b_1 t + b_0$. Then we have

$$\sum_{t\geq 0} f(t)z^t = \sum_{t\geq 0} b_d t^d z^t + \sum_{t\geq 0} b_{d-1} t^{d-1} z^t + \ldots + \sum_{t\geq 0} b_1 t z^t + \sum_{t\geq 0} b_0 z^t$$

From **Problem 2** we proved that

$$\sum_{t\geq 0} (t+1)^d z^t = \frac{A(d,1)z^0 + \cdots + A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

$$z\sum_{t\geq 0} (t+1)^d z^t = \sum_{t\geq 0} (t+1)^d z^{t+1} = \frac{A(d,1)z^1 + \cdots + A(d,d)z^d}{(1-z)^{d+1}}$$

We then have

$$\sum_{t\geq 1} (t)^d z^t = \sum_{t\geq 0} (t)^d z^t = \frac{A(d,1)z^1 + \cdots + A(d,d)z^d}{(1-z)^{d+1}}$$

Using this we have

$$\sum_{t\geq 0} f(t)z^t = b_d \frac{A(d,1)z^1 + \cdots + A(d,d)z^d}{(1-z)^{d+1}} + b_{d-1} \frac{A(d-1,1)z^1 + \cdots + A(d-1,d-1)z^{d-1}}{(1-z)^d}$$

$$+ \ldots + b_1 \frac{A(1,1)z^1}{(1-z)^2} + b_0 \frac{1}{(1-z)^1}$$

Denote $g_k(z) = A(k,1)z^1 + \ldots A(k,k)z^k$, hence $g_k(z)$ is a polynomal of degree k and therefore we have

$$\sum_{t\geq 0} f(t)z^t = \sum_{k=0}^{d} b_k \frac{g_k(z)}{(1-z)^{k+1}}$$

Since we want to add all these together we need a common denominator. Hence

$$\sum_{t\geq 0} f(t)z^t = \sum_{k=0}^{d} b_k \frac{g_k(z)(1-z)^{d-k}}{(1-z)^{d+1}}$$

Thus our numerator is $g(z) = \sum_{k=0}^{d} b_k g_k(z)(1-z)^{d-k}$. This polynomial's degree is at most d and

$$g(z) = \sum_{k=0}^{d} b_k g_k(z)(1-z)^{d-k} + g_k(z)(1-z)^0 = \sum_{k=0}^{d-1} b_k g_k(z)(1-z)^{d-k} + g_d(z)$$

Therefore

$$g(1) = \sum_{k=0}^{d-1} b_k g_k(1)(1-1)^{d-k} + g_d(1) = g_d(1) = A(d,1) + \ldots A(d,d) > 0$$